Explicit solutions for dynamic portfolio choice in jump-diffusion models with multiple risky assets and state variables and their applications

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Abstract: This paper studies the optimal portfolio selection problem in jump-diffusion models where there are potentially a large number assets and/or state variables. More specifically, we derive closed form solution for the optimal portfolio weights up to solving a set of ordinary differential equations (ODEs). First, our results generalize Liu (2007) models by incorporating jumps in both stock returns and state variables. Second, we extend Jin and Zhang (2012) results by including jumps in state variables. To examine the effects of jump on an investor's behavior, we then apply our results to two examples. In the first application, we propose a novel self-exciting jump intensity process in a double jump model and explicitly solve the optimal investments in variance swaps. The second application investigate the impact of jump in stock return on cash-bond-stock mix by revisiting the bond/stock ratio puzzle in a jump-diffusion model.

JEL Classification: G11

Keywords: Optimal portfolio selection, jump-diffusion model, variance swap, bond/stock ratio puzzle

1   Introduction

Mounting empirical evidence suggests that the jump risk needs to be captured in asset price processes and other risk factors, such as volatility processes, in addition to the diffusion risk. For example, Eraker, Johannes and Polson (2003), Eraker (2004), Chernov, Gallant, Ghysels, and Tauchen (2003), among others, find strong evidence for co-jumps in volatility and stock returns, i.e., a big jump in stock prices is likely to be associated with a big jump in volatility. It is well understood that jump risk in stock prices has a substantial impact on portfolio selection. Specifically, in a single-stock jump-diffusion model, Liu et al. (2003) find that an investor is less willing to take leveraged or short positions than in a standard pure-diffusion

model, due to the investor's inability to hedge jump risk in stock price through continuous rebalancing. In an international market setting, Das and Uppal (2004) investigate the effect of systemic jumps in stock prices on international portfolio selection. They find that jumps reduce the gain from international diversification and that leveraged portfolios may incur large losses when jumps occur. Despite the large literature on portfolio choice with jumps in asset prices, there are only a few studies on asset allocation in the presence of jumps in both stock prices and state variables. Liu, Longstaff and Pan (2003) solve the optimal portfolio choice problem in closed form for a model where there is only one risky asset with jumps in both stock price and volatility. In this paper, we solve the optimal asset allocation problem in closed form for multi-asset jump-diffusion models under conditions similar to those in Duffie, Pan and Singleton (2000) and Liu (2007). In contrast to Das and Uppal (2004), an important feature of our model is that both stock prices and state variables are allowed to jump. To the best of our knowledge, we are not aware of closed-form solution for dynamic asset allocation problem in a jump-diffusion model where a risk averse investor faces jumps in multiple risky assets and state variables. More importantly, the explicit solutions greatly facilitate economics insight and empirical applications.

As prompted by the seminal work of Merton (1969, 1971) and Samuelson (1969), there is a large literature on the dynamic portfolio choice problem which has typically been studied with continuous-time models primarily due to their analytical tractability. For pure-diffusion models, see Cox and Huang (1989), Detemple, et al. (2003), Liu (2007); for jump-diffusion models, see Liu, Longstaff and Pan (2003), Das and Uppal (2004), Aït-Sahalia, et al. (2009), Jin and Zhang (2012), and among many others. As is well-understood, it is a daunting task to solve the optimal portfolio choice problem especially in an incomplete market with a large number state variables. For complete pure-diffusion models, based on a refined version of a method developed in Ocone and Karatzas (1991), Detemple, et al. (2003) solve the portfolio choice problem in a pure-diffusion model which may include a large number of assets and state variables with non-affine structures, and they obtain the optimal portfolio strategy by using Monte Carlo simulation. But one of their key assumptions is the completeness of the market and their simulation based approaches may be time-consuming in the presence of a

large number of assets and/or state variables.

It is well-known that by assuming quadratic conditions in pure-diffusion models, Liu (2007) explicitly solve the optimal dynamic portfolio choice problem in both complete and incomplete markets, up to the solution of a set of ordinary differential equations (ODEs). Specifically, he solves a set of ODEs by guessing the exponential linear form of the indirect value function without simulation. In contrast, much less is known about conditions which lead to analytic solution to the optimal portfolio choice in jump-diffusion models especially when both stock prices and state variables are allowed to jump. As it will be clear, jumps pose an obstacle to this popular method when state variables are incorporated. The objective of the present paper is to generalize the ODE-based approach to jump-diffusion models. In particular, we provide conditions under which the indirect value function in jump-diffusion models has the exponential linear form. And then, the indirect value function and the optimal portfolio strategy can be obtained by solving a set of ODEs.

Our paper is closely related to the work of Jin and Zhang (2012) in that they use a decomposition approach based on HJB equation to solve a portfolio selection problem that may include a large number of assets and state variables. But their state variables are pure-diffusion processes and the indirect value function is evaluated by Monte Carlo simulation. Our paper also relates to the work of Das and Uppal (2004) and Aït-Sahalia, et al. (2009). These researchers solve the portfolio selection problems for jump-diffusion models. However, in their models, there is no state variable. In contrast, we obtain closed-form solutions to the optimal portfolio strategies under jump-diffusion models that can include a large number of assets and state variables.

Our explicit solutions allow us to solve in a computationally efficient way the dynamic portfolio selection problem in jump-diffusion models and facilitate insight of an investor's behavior when facing jumps in stock returns and/or state variables. For concreteness, we focus on two applications. The first application is variance swap investment. As is well-understood, the variance swaps provide good investment opportunities as one can trade volatility directly to exploit the volatility risk premium instead of indirectly via trading options. We present a tractable model for solving the optimal portfolio choice problem

in variance swap market where a power-utility investor trades three variance swaps with different maturities and a riskfree bond. More specifically, based on the SV2F-PJ-VJ model in Ait-Sahalia et al. (2015), we propose a new self-exciting process to model the jump intensity because it seems hard to analytically solve the optimal asset allocation problem involving variance swap in SV2F-PJ-VJ model. In contrast to the SV2F-PJ-VJ model in Ait-Sahalia et al. (2015), our model has two attractive features. First, unlike the SV2F-PJ-VJ model in Ait-Sahalia et al. (2015), given two variance swaps, a third variance swap is not redundant in the sense that the variance swap rate of third variance swap cannot be replicated by trading the two given variance swaps. And thus, this makes the third variance swap valuable for investment. Furthermore, any three variance swaps can span the linear space generated by three sources of risks: short-run variance, long-run variance and jump. Second, we obtain closed-form solution to the optimal investment in variance swaps. In calibration exercises, our empirical results show that it is always optimal to take long positions in the medium-term variance swap contracts and short positions in both the short-term and the long-term contracts. This is in stark contrast to the trading positions in the two variance swap contracts in the two-factor pure-diffusion model examined in Egloff, Leippold and Wu (2010) where an investor can take either long or short position in each contract depending on the model parameters. As demonstrated in Zhou and Zhu (2012), the cost of using one-factor Heston stochastic volatility model can be as high as 70%. This brings us to ask what economics cost is by ignoring jumps in volatility. Surprisingly, we find in all examples tested that if our double jump model is the true model, then the strategy obtained from the pure-diffusion model in Egloff, Leippold and Wu (2010) always violate the jump-induced constraint on jump exposure and thus leads to 100 percent wealth equivalent loss by following the suboptimal strategy. In short, our results along with those in Zhou and Zhu (2012) indicate the serious consequence of model misspecification for variance swap investment.

Application 2 examines how jumps in stock return affects the optimal cash-bond-stock mix in a dynamic asset allocation model where an investor can trade one stock, two bonds and cash. We revisit the asset allocation puzzle raised in Canner et al. (1997). Namely, the

empirical evidence documented in their paper shows that strategic asset allocation advices tend to recommend a higher bond/stock ratio for a more risk averse investor. Several authors have attempted to explain the rationality of the puzzle. For instance, Bajeux-Besnainou et al. (2001) and Brennan and Xia (2000) relate the puzzle to a hedging component to a stochastic interest rate and provide an elegant solution to the asset allocation puzzle. More specifically, as pointed by Lioui (2007), the puzzle can be resolved under the assumption that one or several bonds can perfectly hedge the risk from the interest rate and the market price of risk. This approach is extended by Lioui (2007), which demonstrates that the puzzle may be still a puzzle for an bond market where the above hedging assumption is invalid. All of these studies assume that short term interest rate and stock return follow pure diffusion processes. Our framework generalizes these studies by incorporating jumps in stock returns and examining the roles of risk aversion in determining the optimal cash-bond-stock mix. In particular, we show that unlike the pure-diffusion models in Bajeux-Besnainou et al. (2001), Brennan and Xia (2000) and Lioui (2007), there is no clear-cut answer to the bond/stock ratio puzzle in a jump-diffusion model despite the aforementioned hedging assumption. This finding strengthens the claim made by Lioui (2007) that the asset allocation puzzle is still a puzzle.

The rest of the paper is organized as follows. In the next section, we present the framework for Merton's dynamic portfolio selection problem in jump-diffusion models. Then, we present conditions in the the jump-diffusion models under which we explicitly solve indirect value function and the optimal portfolio strategy in terms of solution to a set of ODEs. In Section 3, we propose a new self-exciting process for jump intensity in a double jump-diffusion model and explicitly solve the optimal portfolio weights in a market consisting of one riskfree bond and three variance swaps with different maturities. Finally, in Section 4, we derive closed-form solution to the optimal cash-bond-stock mix and especially investigate how jump risk in stock return affect bond/stock ratio. We conclude in Section 5. All proofs are collected in Appendix.

2 Merton's portfolio choice problem

In this section we formulate a model of incomplete financial markets in a continuous time economy where asset prices and state variables follow a multidimensional jump-diffusion process on the fixed time horizon $[0, T]$, $0 < T < \infty$. We consider a complete probability space $(\Omega, \mathcal{F}, P)$, where Ω is the set of states of nature with generic element ω, $\mathcal{F}$ is the σ-algebra of observable events and P is a probability measure on $(\Omega, \mathcal{F})$.

We use a l-dimensional vector $X_t = (X_{1t}, ..., X_{lt})^\top$ to denote the state variables of the economy where the convention $\top$ stands for the transpose of a vector or a matrix. The state variables X_t may include stochastic volatility and stochastic interest rate as its components. We assume that state variables X_t follow a jump-diffusion process

$$dX_t = b^x(X_t)dt + \sigma^x(X_t)dB^X(t) + \sigma_J^x(X_t)(Y^x \bullet dN(t))$$

where $b^x(X_t)$ is an l-dimensional vector function, $\sigma^x(X_t)$ is an $l \times l$ matrix function of X_t, and $\sigma_J^x(X_t)$ is an $l \times m$ matrix function of X_t, respectively. It should be noted that unlike Liu (2006) and Jin and Zhang (2012), the above specification of X_t includes jumps in state variables. For instance, we can incorporate a jump in volatility process. By letting $Y^x = 0$, our jump-diffusion model reduces to their pure-diffusion models. $B^X(t) = (B_1^X(t), ..., B_l^X(t))^\top$ is a l-dimensional standard Brownian motion; $N(t) = (N_1(t), ..., N_m(t))^\top$ is an m-dimensional multivariate Poisson process with $N_k(t)$ denoting the number of type k jumps up to time t; and $Y^x = (Y_1^x, ..., Y_m^x)^\top$ with Y_k^x denoting the amplitude of the type k jump conditional on the occurrence of the k-th jump. For any two n-dimensional vectors $x = (x_1, ..., x_n)^\top$ and $y = (y_1, ..., y_n)^\top$, we denote the component-wise multiplication as $x \bullet y = (x_1 y_1, ..., x_n y_n)^\top$.

The uncertainty of the economy is also generated by a d-dimensional standard Brownian motion $B^S(t) = (B_1^S(t), ..., B_d^S(t))^\top$, which drives stock prices defined below. Assume $B^S(t)$ and $B^X(t)$ are correlated and $E[dB^X(t)d(B^S(t))^\top] = \rho_t dt$, for some $l \times d$ matrix ρ_t. The flow of information in the economy is given by the natural filtration, i.e., the right-continuous and augmented filtration $\{\mathcal{F}_t\}_{t \in [0,T]} = \{\mathcal{F}_t^S \vee \mathcal{F}_t^X \vee \mathcal{F}_t^N, t \in [0, T]\}$, where $\mathcal{F}_t^S = \sigma(B^S(s); 0 \leq s \leq$

t), $\mathcal{F}_t^X = \sigma(B^X(s); 0 \le s \le t)$ and $\mathcal{F}_t^N = \sigma(N(s); 0 \le s \le t)$. We suppose that observable events are eventually known, i.e., $\mathcal{F} = \mathcal{F}_T$. For illustrative purposes[1], we assume that N_k admits stochastic intensity $\lambda_k(X_t)$, where $\lambda_k(X_t)$ represents the rate of the jump process at time t.

We are now in a position to describe asset price processes. The market considered in this paper includes $n + 1$ assets traded continuously on the time horizon $[0, T]$. One of these assets, risk-free, has a price $S_0(t)$ which evolves according to the differential equation

$$dS_0(t) = S_0(t)r(X_t)dt, \qquad S_0(0) = 1. \tag{1}$$

The remaining n assets, called stocks, are risky; their prices are modeled by the linear stochastic differential equation

$$\frac{dS_i(t)}{S_i(t-)} = b_i(X_t)dt + \sigma_i^b(X_t)dB^S(t) + \sigma_i^q(X_t)(Y^s \bullet dN^S(t))$$

where $i = 1, ..., n$, $N^S(t) = (N_1(t), ..., N_{n-d}(t))^\top$, and $Y^s = (Y_1^s, ..., Y_{n-d}^s)^\top$, with Y_k^s denoting the amplitude of the type k jump conditional on the occurrence of the k-th jump. Here $\sigma_i^b(X_t)$ is the d-dimensional diffusion coefficient row vector and $\sigma_i^q(X_t)$ is the $(n - d)$-dimensional jump coefficient row vector. In particular, the Brownian motions represent frequent small movements in stock prices, while the jump processes represent infrequent large shocks to the market. Assuming $n - d \le m$, the jumps $N^S(t)$ can be regarded as common jumps in stock returns and state variables.

We now turn to the portfolio selection problem. In this paper, we focus on the Merton's problem of maximizing the expected utility from the terminal wealth. For analytic tractability, we consider the constant relative risk aversion (CRRA) utility function given by

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \forall x > 0; \\ -\infty, & \forall x \le 0, \end{cases} \tag{2}$$

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[1]Our results can be extended to infinite activity jump processes.

where γ is the relative risk aversion (RRA) coefficient. Specifically, we consider an investor with the utility function $U(x)$ and endowed with some initial wealth w_0, which is invested in the above-mentioned $n+1$ assets. Let $\pi(t) = (\pi_1(t), ..., \pi_n(t))^\top$ denote a trading strategy, where $\pi_i(t)$ is the proportion of total wealth invested in the i-th risky asset held at time t and $\mathcal{F}_t$-predictable. Any portfolio policy $\pi(t)$ has an associated wealth process W_t that evolves as

$$
\begin{aligned}
W_t &= W_0 + \int_0^t r(s)W_s ds + \int_0^t W_s \pi^\top(s)(b(s) - r(s)\mathbf{1}_n)ds \\
&\quad + \int_0^t W_s \pi^\top(s)\Sigma_b(X_s)dB^S(s) + \int_0^t W_{s-}\pi^\top(s-)\Sigma_q(X_s)(Y^s \bullet dN^S(s))
\end{aligned}
$$

where $b(t) = (b_1(X_t), ..., b_n(X_t))^\top$, $\Sigma_b(X_t)$ is an $n \times d$ matrix with σ_i^b being its i-th row, $\Sigma_q(X_t)$ is the $n \times (n-d)$ matrix, with σ_i^q being its i-th row. Here we use $\mathbf{1}_n$ to denote the n-dimensional column vector of ones. A portfolio rule $\pi(t)$ is said to be admissible if the corresponding wealth process satisfies $W_t \geq 0$ almost surely. We use $\mathcal{A}(w_0)$ to denote the set of all admissible trading strategies. Then, the traditional Merton's problem is that the investor attempts to maximize the following quantity

$$
u(w_0) = \max_{\pi \in \mathcal{A}(w_0)} J(w_0) = E\left[U(W_T)\right].
$$

For illustrative purposes, we assume $n - d = m$ because it is straightforward to extend our results to the case: $n - d < m$. Following Merton (1971), using the standard approach to stochastic control and an appropriate Ito's lemma for jump-diffusion processes, we can derive the optimal portfolio weights, π, and the corresponding indirect value function, J, of the investor's problem following the HJB equation below:

$$
\begin{aligned}
0 = \max_\pi \Bigg\{ &J_t + \frac{1}{2}W^2 \pi^\top \Sigma_b \Sigma_b^\top \pi J_{WW} + W[\pi^\top(b(t) - r\mathbf{1}_m) + r]J_W \\
&+ b^x(t)J_X + W\pi^\top \Sigma_b \rho_t^\top \sigma^{x\top}(t)J_{WX} + \frac{1}{2}Tr(\sigma^x(t)\sigma^{x\top}(t)J_{XX^\top}) \\
&+ \sum_{k=1}^{n-d} E[J(W + W\pi^\top \Sigma_{qk}Y_k^s, Y_k^x) - J(W)] \Bigg\}
\end{aligned} \tag{3}
$$

9

where $\boldsymbol{\Sigma}_{qk}$ denotes the k-th column of $\boldsymbol{\Sigma}_q$. The above HJB equation nests the HJB equation (3) for pure-diffusion model in Liu (2006) as a special case by letting $n - d = 0$. It is well-known that in the pure-diffusion model in Liu (2006), the indirect value function $J(t, W, X_t)$ is conjectured to have the form: $J(t, W, X_t) = \frac{W^{1-\gamma}}{1-\gamma} \left[e^{A(t)+B(t)X_t} \right]^\gamma$, where $A(t)$ is a scalar and $B(t)$ is a $1 \times l$ vector. Then, under the quadratic conditions, a set of ODEs for the functions $A(t)$ and $B(t)$ is obtained by substituting the function J into the HJB equation (3). As shown below, the argument in Liu (2006) does not trivially apply to jump-diffusion models. More specifically, compared with the HJB equation (3) in Liu (2006) for pure-diffusion model, the last term in the above HJB equation is a new term due to the presence of jumps. More importantly, this jump term creates new difficulties for closed-form solutions to the optimal portfolio choice problem in the jump-diffusion models. To illustrate these difficulties specific to the jump-diffusion model, we consider a simple case where there are no jumps in the state variables X_t by letting $Y_k^x = 0, k = 1, ..., n - d$. By following the literature, we substitute the indirect value function $J(t, W, X_t) = \frac{W^{1-\gamma}}{1-\gamma} (f(t, X_t))^\gamma$ into (3) and get the following form for the last term:

$$
\sum_{k=1}^{n-d} E[J(W + W\pi^\top \boldsymbol{\Sigma}_{qk} Y_k^s, Y_k^x) - J(W)]
$$
$$
= \frac{W^{1-\gamma}}{1-\gamma} (f(t, X_t))^\gamma \sum_{k=1}^{m} \lambda_k(X_t) E[(1 + \pi^\top \boldsymbol{\Sigma}_{qk} Y_k^s)^{1-\gamma} - 1].
$$

As is well-understood from, for instance, Liu (2006), in order to get an explicit solution for the indirect value function $J(t, W, X_t)$ of the form $J(t, W, X_t) = \frac{W^{1-\gamma}}{1-\gamma} \left[e^{A(t)+B(t)X_t} \right]^\gamma$, the term $E[(1 + \pi^\top \boldsymbol{\Sigma}_{qk} Y_k^s)^{1-\gamma}]$ should be an affine function of the state variables X_t. This term, however, is hard to be an affine function of the state variables X_t unless the optimal jump exposure $\pi^\top \boldsymbol{\Sigma}_{qk}$ is a deterministic function of time t. The reason for this is that the function $x^{1-\gamma}$ is generally not an affine function. Based on this observation and inspired by the results in Liu (2006) and the result of decomposition of optimal portfolio weights in Jin and Zhang (2012), we are able to specify an affine model[2], which leads to ODEs for $A(t)$ and $B(t)$ given

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[2]Here, for expositional purposes, we consider affine models only as it is straightforward to generalize our results to quadratic processes defined in Liu (2006).

in Proposition 1 below.

To this purpose, we now introduce more notations. We let $a_k = E(Y_k^s), k = 1, ..., n - d$. We assume the matrix $\Sigma = [\Sigma_b, \Sigma_q]$ is invertible. The market price of risk is represented by

$$\begin{pmatrix} \theta^b \\ \theta^q \end{pmatrix} = \Sigma^{-1}(b(t) - r\mathbf{1}_n + \Sigma_q(\lambda \bullet a)), \tag{4}$$

where $\lambda \bullet a = (\lambda_1 a_1, ..., \lambda_{n-d} a_{n-d})^\top, \theta^b = (\theta_1^b, ..., \theta_d^b)^\top$ and $\theta^q = (\theta_1^q, ..., \theta_{n-d}^q)^\top$. As will become clear in next sections, θ_i^b is the risk premium for the Brownian motion $B_i^S, i = 1, ..., d$, and θ_k^q represents the risk premium for the jump $N_k^S, k = 1, ..., n - d$, in the stock returns. We now make the following assumptions:

$$
\begin{aligned}
b^x(X_t) &= k - KX, \\
\sigma^x \sigma^{x\top} &= h_0 + h_1 \cdot X, \\
r &= \delta_0 + \delta_1^\top X, \\
\theta^{b\top} \theta^b &= H_0 + H_1^\top X, \\
\sigma^x \rho_t \theta^b &= g_0 + g_1 X, \\
\sigma^x \rho_t \rho_t^\top \sigma^{x\top} - \sigma^x \sigma^{x\top} &= l_0 + l_1 \cdot X, \\
\lambda &= \lambda_0 + \lambda_1 X, \\
\theta_k^q &= \theta_k^0 \lambda_k, k = 1, ..., n - d,
\end{aligned}
$$

where k, δ_1, H_1 and g_0 are $l \times 1$ constant vectors, K, h_0, g_1 and l_0 are $l \times l$ constant matrices, δ_0, H_0 and θ_k^0 are constants, λ_0 is a $(n - d) \times 1$ constant vector, λ_1 is a $(n - d) \times l$ constant matrix, $h_1 = h_{1jk}^i, i, j, k = 1, ..., l$ and $l_1 = l_{1jk}^i, i, j, k = 1, ..., l$ are constant tensors with three indices (one upper index and two lower indices). In particular, $h_1 \cdot X$ is a $l \times l$ matrix whose (j, k) element is

$$(h_1 \cdot X)_{jk} = \sum_{i=1}^{l} h_{1jk}^i X_{it}.$$

The $l \times l$ matrix $l_1 \cdot X$ is defined in exactly the same manner. The above assumptions except the last two are similar to those made in Liu (2007); the last two assumptions on jump intensity and jump premium are also standard ones made in literature, in particular, the last assumption says that the jump risk premium for the k-th jump is proportional to its intensity.

As demonstrated in Liu and Pan (2003), Zhou (2012), Jin and Zhang (2012), it seems more convenient to first solve the optimal diffusion exposure $\widetilde{\pi}_b^* = (\pi^\top \Sigma_b)^*$ and jump exposure $\widetilde{\pi}_q^* = (\pi^\top \Sigma_b)^*$, and then find the optimal portfolio weights according to the formula in Proposition 2 given below. For this, let us define the following vector

$$D = (D_1, ..., D_{n-d}),$$

where

$$D_k = (1 - \gamma)\widetilde{\pi}_{qk}^*(\theta_k^0 - a_k) + E^P\left[\left(\widetilde{\pi}_{qk}^* Y_k^s + 1\right)^{1-\gamma} e^{\gamma B \sigma_{Jk}^x Y_k^x} - 1\right], k = 1, ..., n - d,$$

where $\widetilde{\pi}_{qk}^*$ is the optimal exposure to the k-th jump and is determined in Proposition 2 below.

Proposition 1 *Under the above assumptions, we have the following results:*

$$J(t, W_t, X_t) = \frac{W_t^{1-\gamma}}{1-\gamma}[f(t, X_t)]^\gamma = \frac{W^{1-\gamma}}{1-\gamma}\left[e^{A(t)+B(t)X_t}\right]^\gamma \tag{5}$$

where the functions $A(t)$ and $B(t)$ satisfy the following equations:

$$\frac{dA}{dt} + \left(k + \frac{1-\gamma}{\gamma}g_0\right)^\top B^\top + \frac{1}{2}B[h_0 + (1-\gamma)l_0]B^\top$$
$$+ \frac{1-\gamma}{2\gamma^2}H_0 + \frac{1-\gamma}{\gamma}\delta_0 + \lambda_0^\top D^\top = 0,$$
$$\frac{dB}{dt} + \left(-K + \frac{1-\gamma}{\gamma}g_1\right)^\top B^\top + \frac{1}{2}B[h_1 + (1-\gamma)l_1]B^\top$$
$$+ \frac{1-\gamma}{2\gamma^2}H_1 + \frac{1-\gamma}{\gamma}\delta_1 + \lambda_1^\top D^\top = 0$$

Proof. See Appendix A. ∎

The following result presents the optimal portfolio weights.

Proposition 2 *The optimal portfolio weight $\pi^* = (\pi_1^*, ..., \pi_n^*)$ is given by*

$$\pi^* = \left(\widetilde{\pi}_{b1}^*, ..., \widetilde{\pi}_{bd}^*, \widetilde{\pi}_{q1}^*, ..., \widetilde{\pi}_{q(n-d)}^*\right) \Sigma^{-1} \tag{6}$$

where

$$\left(\widetilde{\pi}_{b1}^*, ..., \widetilde{\pi}_{bd}^*\right)^\top = \frac{\widetilde{\theta}^b}{\gamma} + \rho_t \sigma^{x\top} B^\top(t)$$

and $\widetilde{\pi}_{qk}^*$ solves the following optimization problem:

$$\sup_{\widetilde{\pi}_{qk} \in F_k} \widetilde{\pi}_{qk}(\theta_k^0 - a_k) + \frac{1}{1-\gamma} \int_{A_k} [(1 + \widetilde{\pi}_{qk}z)]^{1-\gamma}\Phi_k(dz) \tag{7}$$

for $k = 1, ..., n - d$, where A_k denotes the support of k-th jump size Y_k^s and F_k is the set of feasible k-th jump exposures satisfying the no-bankruptcy condition, namely, $F_k = \{x | x \cdot y > -1, \forall y \in A_k\}$.

Proof. See Appendix A. ∎

The objective function in optimization problem (7) does not include the state variables X_t and thus, for each k, the optimal jump exposure $\widetilde{\pi}_{qk}^*$ is deterministic. This justifies the conjectured exponential linear form of the indirect value function. It is worth mentioning that despite the deterministic jump exposure $\widetilde{\pi}_{qk}^*$, the optimal portfolio weights π^* is still dependent of the state variables X_t through the optimal diffusion exposures $(\widetilde{\pi}_{b1}^*, ..., \widetilde{\pi}_{bd}^*)$ and the matrix Σ. For example, considering the quadratic process for the state variables X_t in Liu (2007), the optimal diffusion exposures $(\widetilde{\pi}_{b1}^*, ..., \widetilde{\pi}_{bd}^*)$ is a linear function of the state variables and thus, from (6), the optimal portfolio weights π^* is linear in the state variables too if Σ is a constant matrix. This state-dependent portfolio strategy reflects the investor's market timing behavior.

3 Dynamic asset allocation for variance swaps

In order to exploit variance risk premium and hedge variance risk, the variance swap contract has become the most actively traded variance-related derivative security due to its

direct exposure to volatility and provide good investment opportunity [3]. Despite the growing importance of trading variance swap, surprisingly, there are only few papers available studying dynamic portfolio choice problem incorporating variance swap. For example, in a two-factor affine pure-diffusion model, Egloff, Leippold and Wu (2010) explicitly solves the optimal dynamic portfolio choice problem and finds that it is optimal for an investor to take a short position in a short-term variance swap and a long position in a long-term variance swap. In contrast, Filipovié, Gourier and Mancini (2015) show that an investor optimally takes a short position in a long-term variance swap to earn the significant negative variance risk premium and a long position in a short-term variance swap to hedge the volatility risk. In fact, Filipovié, Gourier and Mancini (2015) study the optimal portfolio choice problem involving variance swap in quadratic pure-diffusion variance swap models and hence closed-form solution as in Egloff, Leippold and Wu (2010) is unavailable. In particular, in the pure-diffusion models of both Egloff, Leippold and Wu (2010) and Filipovié, Gourier and Mancini (2015), only two variance swaps are incorporated in the portfolio choice problem because any two variance swaps can span the two sources of risk in two-factor variance swap rate dynamics and thus a third variance swap is redundant.

As widely documented strong empirical evidence, see, for example, Broadie, Chernov and Johannes (2007), Todorov (2009), Bandi and Reno (2015), suggests that both stock return and volatility exhibit jumps and jumps play a key role in explaining the observed risk premium, Aït-Sahalia, Karaman and Mancini (2015) extend the two-factor model in Egloff, Leippold and Wu (2010) by incorporating jumps in both stock return and volatility. Interestingly, like the aforementioned pure-diffusion models, given two traded variance swaps, a third variance swap is still redundant and thus variance swaps fail to span the linear space generated by three sources of risk (two diffusions and one jump). The reason for this is that in the "SV2F-PJ-VJ" model in Aït-Sahalia, Karaman and Mancini (2015), the intensity of jump in variance process is an affine function of variance.[4] As a result, given any two variance

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[3]As indicated by Table 7 in Egloff, Leippold and Wu (2010), the investment strategies involving variance swap have a Sharpe ratio of at least 1.20 while the Sharpe ratio of S&P is at most 0.5.

[4]This modeling also appears to make it hard to obtain an analytic solution to the optimal portfolio choice problem incorporating variance swap.

swaps available for trading, an investor does not benefit from including a third variance swap in her portfolio. Importantly, the redundance makes it hard to solve the optimal variance swap investment problem in closed form.

The above observations bring us to ask the following questions: If the link between the jump intensity and variance in Aït-Sahalia, Karaman and Mancini (2015) is invalid, are two variance swaps sufficient to span the variance swap rate process? If not, what does an optimal strategy involving three variance swaps look like as opposed to the short-long rule in Egloff, Leippold and Wu (2010) In the meantime, what is the benefit from trading three variance swaps? To address these questions, the primary objective of this section is to study optimal investment in variance swap in double jump models. More specifically, by virtue of the results in Propositions 1 and 2, we exploit the problem of optimal variance swap investment in a analytically tractable double jump model which modifies the jump intensity of the "SV2F-PJ-VJ" model in Aït-Sahalia, Karaman and Mancini (2015). First, a general model in the context of Section 2 is specified, and its properties are discussed and numerically compared to other models that are widely employed in the literature. Finally, analytic solutions to the problem of dynamic asset allocation for variance swaps are provided and their performance is examined in the numerical examples. Hence, our study contributes to the literature by enhancing our understanding of variance swap trading when both stock price and volatility can jump.

3.1 Model Specification and Properties

For analytic tractability, we adopt the popular double jump model used by Aït-Sahalia, Karaman and Mancini (2015), apart from the specification for the intensity of the counting process. That is, we assume that the stock price, volatility and its long-run mean under a risk-neutral measure Q are given as follows:

$$
\begin{aligned}
\frac{dS_t}{S_{t-}} &= (r - \delta)dt + \sqrt{(1 - \rho^2)v_t}dW_{1t}^Q + \rho\sqrt{v_t}dW_{2t}^Q + (\exp(J_t^{s,Q}) - 1)dN_t - g^Q\lambda_t dt, \\
dv_t &= \kappa_v^Q(m_t - v_t)dt + \sigma_v\sqrt{v_t}dW_{2t}^Q + J_t^{v,Q}dN_t \\
dm_t &= \kappa_m^Q(\theta_m^Q - m_t)dt + \sigma_m\sqrt{m_t}dW_{3t}^Q.
\end{aligned}
\tag{8}
$$

As in Aït-Sahalia, Karaman and Mancini (2015), we specify the market price of risks for the Brownian motions by γ_i ($i = 1, 2, 3$) in the following way:

$$\Lambda_t^\top = [\gamma_1\sqrt{(1-\rho^2)v_t}, \gamma_2\sqrt{v_t}, \gamma_3\sqrt{m_t}],$$

Then, under the objective probability P, the stock price and variance dynamics can be represented as follows:

$$\frac{dS_t}{S_{t-}} = \mu_t dt + \sqrt{(1-\rho^2)v_t}dW_{1t}^P + \rho\sqrt{v_t}dW_{2t}^P + (\exp(J_t^{s,P}) - 1)dN_t - g^P\lambda_t dt,$$

$$dv_t = \kappa_v^P(m_t\kappa_v^Q/\kappa_v^P - v_t)dt + \sigma_v\sqrt{v_t}dW_{2t}^P + J_t^{v,P}dN_t \tag{9}$$

$$dm_t = \kappa_m^P(\theta_m^P - m_t)dt + \sigma_m\sqrt{m_t}dW_{3t}^P,$$

where $\mu_t = r - \delta + \gamma_1(1-\rho^2)v_t + \gamma_2\rho v_t + (g^P - g^Q)\lambda_t$, $\kappa_v^P = \kappa_v^Q - \gamma_2\sigma_v$, $\kappa_m^P = \kappa_m^Q - \gamma_3\sigma_m$, and $\theta_m^P = \theta_m^Q\kappa_m^Q/\kappa_m^P$, while r is the risk free rate, and δ is the dividend yield, both taken to be constant for simplicity. The correlation parameter ρ is used to capture the so-called leverage effect between stock returns and variance changes. The three Brownian motions, $W_{it}^Q, i = 1, 2, 3$, are uncorrelated.[5]

The dynamics of the spot variance of the price, v_t, is driven by a two-factor model, while the speed of mean revision is κ^P under P (κ^Q under Q accordingly). The long-term mean of the variance is governed by the pure-diffusion process m_t that has a similar specification with v_t but equipped with a parameter triple of κ_m^P (κ_m^Q), θ_m^P (θ_m^Q) and σ_m, respectively. As a result, the process v_t presents the fast mean reverting and volatile pattern and captures sudden movements in variance with the jump process, while the process m_t has no jump and is less volatile and persistent and characterizes the central tendency of variance.

Meanwhile, the jump size in the stock price, $J^{s,Q}$, is independent of both Brownian and jump components, and is assumed to follow a normal distribution with mean μ_j^Q and variance σ_j^2 so that $g^Q = \exp(\mu_j^Q + \sigma_j^2/2) - 1$. Similarly, we may have $g^P = \exp(\mu_j^P + \sigma_j^2/2) - 1$ under

---

[5]The variant of the specifications in Model (8) and (9) is widely used in the literature (see Bakshi, Cao and Chen (1997), Chernov and Ghysels (2000), Bates (2000,2006), Pan (2002), Eraker, Johannes and Polson (2003), Broadie, Chernov and Johannes (2007), Egloff, Leippold and Wu (2010) and Todorov (2009) and references therein).

the objective probability measure P. However, the jump size in the spot variance, $J^{v,Q}$, is positive. It is independent of Brownian motions and the jump component in the stock price, and follows an exponential distribution with parameter μ_v^Q, i.e., $E^Q[J^{v,Q}] = \mu_v^Q$, and so is the jump size $J^{v,P}$. This specification thus captures sudden upward movement of v_t. Furthermore, the two-factor model studied by Pan (2002) that allows for jumps only in stock price can be obtained if both μ_j^P and μ_j^Q are set as zero (e.g., $\mu_j^P = \mu_j^Q \equiv 0$).

We now turn to modeling jump intensity under the measures P and Q, respectively. Empirical studies suggest that the jump intensity of asset prices is stochastic and clustered in time (see Bates (2006) and Aït-Sahalia, Cacho-Diaz and Laeven (2010)). We then assume that the jump intensity λ_t of the counting process N_t under the measure Q follows a self-exciting process as follows:

$$d\lambda_t = \alpha(\lambda_\infty - \lambda_t)dt + \beta_0 J_t^{v,Q}dN_t, \tag{10}$$

where α, λ_∞ and $\beta_0 > 0$.[6] Unlike the the specification of λ_t in Aït-Sahalia, Karaman and Mancini (2015) in the form of $\lambda_t = \lambda_0 + \lambda_1 v_t$, implying that the jump intensity is uniquely determined by volatility, Equation (10) suggests that a jump in either price or variance may cause the intensities to jump up, governing by β_0 and the jump intensity decays exponentially back towards a level λ_∞ at speed α. It partially disentangles the jump intensity from volatility in the sense that λ_t is proportional to v_t when v_t has large movements driven by the jump instead of small one caused by the diffusion.

More importantly, our model is especially tractable in that we solve the optimal portfolio choice problem with variance swap in closed form[7]. Although the specification in Aït-Sahalia, Karaman and Mancini (2015) allows for more jumps to occur during volatile periods with the intensity bounded by a positive constant ($\lambda_0 > 0$), it is subject to the underestimation of volatility in the long run. That is, the mean-reverting nature of volatility suggests that

---

[6]Accordingly, its dynamics under the measure P can be represented as $d\lambda_t = \alpha(\lambda_\infty - \lambda_t)dt + \beta_0 J_t^{v,P}dN_t$.

[7]Our model is also tractable for pricing European options as it is one of affine models developed by Duffie, Pan and Singleton (2000). More recently, Fulop, Li and Yu (2015) propose a self-exciting asset pricing model that takes into account co-jumps between prices and volatility and self-exciting jump clustering. They find that the self-exciting jump intensity has become more important since the onset of the 2008 global financial crisis and illustrate good model performance for the S&P 500 index option data.

the long-term expected volatility declines over time, implying that the intensity of jumps is also a decreasing function of time. This in fact imposes an unnecessary restriction on the dynamics of jump intensity λ_t.

3.2 Term Structure of Variance Swap and Risk Premia

Similar to the equation (9) in Aït-Sahalia, Karaman and Mancini (2015), the variance swap rate is given by

$$VS_{t,t+\tau} = \frac{1}{\tau} \int_t^{t+\tau} v_u du + \frac{1}{\tau} \sum_{u=N_t}^{N_{t+\tau}} (J_u^s)^2 = \bar{v}_{t,t+\tau}^Q + E_t^Q[(J^s)^2]\bar{\lambda}_{t,t+\tau}^Q,$$

where $E_t^Q[(J^{s,Q})^2] = (\mu_j^Q)^2 + \sigma_j^2$,

$$E_t^Q[\lambda_s] = \frac{\alpha \lambda_\infty}{(\alpha - \beta^Q)} \left[1 - e^{-(\alpha - \beta^Q)(s-t)}\right] + \lambda_t e^{-(\alpha - \beta^Q)(s-t)},$$

with $\beta^Q = \beta_0 E_t^Q[J_t^{v,Q}]$.

$$\bar{\lambda}_{t,t+\tau}^Q = \frac{1}{\tau} \int_t^{t+\tau} E_t^Q[\lambda_s] ds = \frac{\alpha \lambda_\infty}{(\alpha - \beta^Q)} \left[1 - \frac{1}{\tau(\alpha - \beta^Q)}(1 - e^{-(\alpha - \beta^Q)\tau})\right]$$
$$+ \frac{\lambda_t}{\tau(\alpha - \beta^Q)}(1 - e^{-(\alpha - \beta^Q)\tau}).$$

Likewise,

$$E_t^Q[m_s] = \theta_m^Q \left[1 - e^{-\kappa_m^Q(s-t)}\right] + m_t e^{-\kappa_m^Q(s-t)},$$

and

$$
\begin{aligned}
E_t^Q[v_s] &= \theta_m^Q \left[1 + \frac{\kappa_m^Q}{\kappa_v^Q - \kappa_m^Q} e^{-\kappa_v^Q(s-t)} - \frac{\kappa_v^Q}{\kappa_v^Q - \kappa_m^Q} e^{-\kappa_m^Q(s-t)} \right] \\
&+ \frac{\alpha \mu_v^Q \lambda_\infty}{\kappa_v^Q(\alpha - \beta^Q)} \left[1 + \frac{\alpha - \beta^Q}{\kappa_v^Q - (\alpha - \beta^Q)} e^{-\kappa_v^Q(s-t)} - \frac{\kappa_v^Q}{\kappa_v^Q - (\alpha - \beta^Q)} e^{-(\alpha - \beta^Q)(s-t)} \right] \\
&+ \frac{\kappa_v^Q}{\kappa_v^Q - \kappa_m^Q} \left[e^{-\kappa_m^Q(s-t)} - e^{-\kappa_v^Q(s-t)} \right] m_t \\
&+ \frac{\mu_v^Q}{\kappa_v^Q - (\alpha - \beta^Q)} \left[e^{-(\alpha - \beta)(s-t)} - e^{-\kappa_v^Q(s-t)} \right] \lambda_t \\
&+ e^{-\kappa_v^Q(s-t)} v_t.
\end{aligned}
$$

Thus, under the risk neutral probability Q, the rate of a variance swap contract with the life time of τ, staring from time t, can be specified as follows:

$$
\begin{aligned}
VS_{t,t+\tau} &= \frac{1}{\tau} \int_t^{t+\tau} E_t^Q[v_s] ds + \frac{1}{\tau} E_t^Q[(J^s)^2] \int_t^{t+\tau} E_t^Q[\lambda_s] ds \\
&= \phi_\theta(\tau)\theta_m^Q + \phi_\lambda^0(\tau)\lambda_\infty + \phi_v(\tau)v_t + \phi_m(\tau)m_t + \phi_\lambda(\tau)\lambda_t,
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_\theta(\tau) &= 1 + \frac{\kappa_m^Q}{\kappa_v^Q \tau(\kappa_v^Q - \kappa_m^Q)} \left[1 - e^{-\kappa_v^Q \tau} \right] - \frac{\kappa_v^Q}{\kappa_m^Q \tau(\kappa_v^Q - \kappa_m^Q)} \left[1 - e^{-\kappa_m^Q \tau} \right], \\
\phi_\lambda^0(\tau) &= \frac{\alpha \mu_v^Q}{\kappa_v^Q(\alpha - \beta^Q)} \left[1 + \frac{(\alpha - \beta^Q)(1 - e^{-\kappa_v^Q \tau})}{\kappa_v^Q \tau(\kappa_v^Q - (\alpha - \beta^Q))} - \frac{\kappa_v^Q(1 - e^{-(\alpha - \beta^Q)\tau})}{(\alpha - \beta^Q)\tau(\kappa_v^Q - (\alpha - \beta^Q))} \right] \\
&+ \frac{\alpha E_t^Q[(J^{s,Q})^2]}{(\alpha - \beta^Q)} \left[1 - \frac{1}{\tau(\alpha - \beta^Q)} (1 - e^{-(\alpha - \beta^Q)\tau}) \right], \\
\phi_v(\tau) &= \frac{1}{\kappa_v^Q \tau} (1 - e^{-\kappa_v^Q \tau}), \\
\phi_m(\tau) &= \frac{\kappa_v^Q}{\tau(\kappa_v^Q - \kappa_m^Q)} \left[\frac{1 - e^{-\kappa_m^Q \tau}}{\kappa_m^Q} - \frac{1 - e^{-\kappa_v^Q \tau}}{\kappa_v^Q} \right], \\
\phi_\lambda(\tau) &= \frac{\mu_v^Q}{\tau(\kappa_v^Q - (\alpha - \beta^Q))} \left[\frac{1 - e^{-(\alpha - \beta^Q)\tau}}{\alpha - \beta^Q} - \frac{1 - e^{-\kappa_v^Q \tau}}{\kappa_v^Q} \right] + \frac{E_t^Q[(J^{s,Q})^2](1 - e^{-(\alpha - \beta^Q)\tau})}{\tau(\alpha - \beta^Q)}.
\end{aligned}
$$

$$(11)$$

Note that given τ, the variance swap rate $VS_{t,t+\tau}$ is a martingale under Q-measure.

Hence, under the objective probability P, $VS_{t,t+\tau}$ follows the equation below

$$
\begin{aligned}
dVS_{t,t+\tau} = &[\phi_v(\tau)\sigma_v\gamma_2 v_t + \phi_m(\tau)\sigma_m\gamma_3 m_t - (\phi_v(\tau) + \beta_0\phi_\lambda(\tau))\mu_v^Q\lambda_t]dt \\
&+ \phi_v(\tau)\sigma_v\sqrt{v_t}dW_{2t}^P + \phi_m(\tau)\sigma_m\sqrt{m_t}dW_{3t}^P + (\phi_v(\tau) + \beta_0\phi_\lambda(\tau))J_t^{v,P}dN_t
\end{aligned}
\tag{12}
$$

3.3   Optimal Variance Swap Allocation

Before examining empirical performance of our model, we solve the optimal portfolio choice problem with variance swap in a model where a risk averse investor can only trade three variance swap and a money market account. [8] In particular, we will show the reason that a third variance swap is redundant in the model of Aït-Sahalia, Karaman and Mancini (2015). As in Egloff, Leippold and Wu (2010), Jin and Zhang (2012), we assume that at time t, the investor initiates three new variance swap contracts with the delivery prices set to be the prevailing variance swap rates $K_1 = VS_{t,t+\tau_1}$, $K_2 = VS_{t,t+\tau_2}$ and $K_3 = VS_{t,t+\tau_3}$. Thus her wealth W_t can be written as

$$
W_t = W_t^M + W_{1t}(VS_{t,t+\tau_1} - K_1) + W_{2t}(VS_{t,t+\tau_2} - K_2) + W_{3t}(VS_{t,t+\tau_3} - K_3),
$$

---

[8]The reason for excluding the stock is that incorporating the stock will introduce two more sources of risk: the diffusion W_{1t} and the jump N_t with jump size $J_t^{s,P}$. In essence, the jump in stock price and the jump in volatility are considered as two different jumps although they occur simultaneously because the two jumps have different random jump sizes. As a result, in order to deliver closed-form solution to the optimal portfolio choice problem when the investor can invest in the stock, we need to incorporate a new asset in addition to the stock. The price of the new asset is driven by the diffusion W_{1t}. For this, we can extend the stock price model in Section 2.3 of Liu (2007) by incorporating jump in the stock price. Specifically, the stock price is

$$
\frac{dS_t}{S_{t-}} = (r_t + \gamma_2\rho v_t + \gamma_1 r_t)dt + \sigma_r\sqrt{r_t}dW_{1t}^P + \rho\sqrt{v_t}dW_{2t}^P + (\exp(J_t^{s,P}) - 1)dN_t - g^P\lambda_t dt,
$$

where r_t is the short rate. And then the investor is allowed to trade a zero-coupon bond, the stock and three variance swaps. In this model, the variance swap rate includes a new term $R_t = E^Q\left[\exp\left(-\int_t^T r_s ds\right)\int_t^T r_s ds\right]$. The dynamics of the expectation can be explicitly derived by using the methods in Duffie, Pan and Singleton (2000) for an affine short rate process. In particular, by adopting the Vasicek model for the short rate r_t, then we can solve the optimal portfolio choice problem in the ODE-based closed form. If r_t is modeled by the CIR process, then, unlike the previous case, the optimal portfolio choice problem can be solved by combining the simulation-based method in Jin and Zhang (2012) and the ODE-based approach in the present paper. A noteworthy feature of the new model is that we can study how the interest rate affect the variance swap rate due to the presence of the term R_t and the investor's demands for the stock, the bond and variance swap. Allowing the investor to access both stock and bond in addition to variance swap will certainly enrich the analysis. We leave this extension as future research.

where W_t^M denotes the amount of money invested in the money market account, W_{1t}, W_{2t} and W_{3t} denote the dollar notional amount invested in the three variance swaps, respectively. As a result, we can write the wealth dynamics as

$$\frac{dW_t}{W_t} = r_t dt + w_{1t} dVS_{t,t+\tau_1} + w_{2t} dVS_{t,t+\tau_2} + w_{3t} dVS_{t,t+\tau_3} \tag{13}$$

where w_{1t}, w_{2t} and w_{2t} denote the fractions of wealth in the three variance swaps, respectively. Plugging the equations for variance swaps into Equation (13), we can recast the wealth dynamics as

$$
\begin{aligned}
\frac{dW_t}{W_t} =\ & r_t dt + w_{1t}[\phi_v(\tau_1)\sigma_v\gamma_2 v_t + \phi_m(\tau_1)\sigma_m\gamma_3 m_t - (\phi_v(\tau_1) + \beta_0\phi_\lambda(\tau_1))\mu_v^Q\lambda_t]dt \\
& + w_{2t}[\phi_v(\tau_2)\sigma_v\gamma_2 v_t + \phi_m(\tau_2)\sigma_m\gamma_3 m_t - (\phi_v(\tau_2) + \beta_0\phi_\lambda(\tau_2))\mu_v^Q\lambda_t]dt \\
& + w_{3t}[\phi_v(\tau_3)\sigma_v\gamma_2 v_t + \phi_m(\tau_3)\sigma_m\gamma_3 m_t - (\phi_v(\tau_3) + \beta_0\phi_\lambda(\tau_3))\mu_v^Q\lambda_t]dt \\
& + w_{1t}[\phi_v(\tau_1)\sigma_v\sqrt{v_t}dW_{2t}^P + \phi_m(\tau_1)\sigma_m\sqrt{m_t}dW_{3t}^P + (\phi_v(\tau_1) + \beta_0\phi_\lambda(\tau_1))J_t^{v,P}dN_t] \\
& + w_{2t}[\phi_v(\tau_2)\sigma_v\sqrt{v_t}dW_{2t}^P + \phi_m(\tau_2)\sigma_m\sqrt{m_t}dW_{3t}^P + (\phi_v(\tau_2) + \beta_0\phi_\lambda(\tau_2))J_t^{v,P}dN_t] \\
& + w_{3t}[\phi_v(\tau_3)\sigma_v\sqrt{v_t}dW_{2t}^P + \phi_m(\tau_3)\sigma_m\sqrt{m_t}dW_{3t}^P + (\phi_v(\tau_3) + \beta_0\phi_\lambda(\tau_3))J_t^{v,P}dN_t].
\end{aligned}
$$

Next result gives the indirect value function.

Proposition 3 *Under the above assumptions, we have the following result:*
$$J(t, W_t, X_t) = \frac{W_t^{1-\gamma}}{1-\gamma}[f(t, X_t)]^\gamma = \frac{W^{1-\gamma}}{1-\gamma}\left[e^{A(t)+B_1(t)v_t+B_2(t)m_t+B_3(t)\lambda_t}\right]^\gamma \tag{14}$$
where the functions $A(t)$, $B(t) = (B_1(t), B_2(t))^\top$ and $B_3(t)$ satisfy the following equations:

$$
\begin{aligned}
\frac{dA}{dt} &+ \kappa_m^P\theta_m^P B_2 + \alpha\lambda_\infty B_3 + \frac{1-\gamma}{\gamma}r = 0, \\
\frac{dB_1}{dt} &- \left(\kappa_v^P - \frac{1-\gamma}{\gamma}\sigma_v\gamma_2\right)B_1 + \frac{1}{2}\sigma_v^2 B_1^2 + \frac{1-\gamma}{2\gamma^2}\gamma_2^2 = 0, \\
\frac{dB_2}{dt} &+ \kappa_v^Q B_1 - \left(\kappa_m^P - \frac{1-\gamma}{\gamma}\sigma_m\gamma_3\right)B_2 + \frac{1}{2}\sigma_m^2 B_2^2 + \frac{1-\gamma}{2\gamma^2}\gamma_3^2 = 0, \\
\frac{dB_3}{dt} &- \alpha B_3 + \frac{\gamma-1}{\gamma}\widetilde{\pi}_{q1}^* E^Q[J^{v,Q}] + \frac{1}{\gamma}E^P\left[\left(\widetilde{\pi}_{q1}^* J^{v,P} + 1\right)^{1-\gamma}e^{\gamma(B_1+B_3\beta_0)J^{v,P}} - 1\right] = 0,
\end{aligned}
$$

21

with $A(T) = B_1(T) = B_2(T) = B_3(T) = 0$.

Proof. See Appendix B. ∎

It is interesting to note that the indirect value function $J(t, W_t, X_t)$ is independent of the maturities τ_1, τ_2 and τ_3 of the three variance swaps because the functions $A(t)$, $B(t) = (B_1(t), B_2(t))^\top$ and $B_3(t)$ do not depend on the three maturities. This can be seen from the above ordinary differential equations satisfied by the for function. In other words, for a CRRA investor, as long as there are three different variance swaps for trading, the maturities of variance swap are irrelevant for her investment performance measured by the indirect value function. This conclusion also holds true in the pure-diffusion model in Egloff, Leippold and Wu (2010) as shown in their Proposition 3. In essence, any two variance swaps with different maturities can span two diffusions in the two-factor pure-diffusion model Egloff, Leippold and Wu (2010) while, as indicated by the nonsingular matrix Σ below, any three variance swaps with different maturities can span the three sources of risk in variance swap rate. In contrast, the optimal portfolio weights depend on the maturities of three variance swaps shown below. To this purpose, we let

$$
\Sigma = \begin{pmatrix}
\phi_v(\tau_1)\sigma_v\sqrt{v_t} & \phi_m(\tau_1)\sigma_m\sqrt{m_t} & \phi_v(\tau_1) + \beta_0\phi_\lambda(\tau_1) \\
\phi_v(\tau_2)\sigma_v\sqrt{v_t} & \phi_m(\tau_2)\sigma_m\sqrt{m_t} & \phi_v(\tau_2) + \beta_0\phi_\lambda(\tau_2) \\
\phi_v(\tau_3)\sigma_v\sqrt{v_t} & \phi_m(\tau_3)\sigma_m\sqrt{m_t} & \phi_v(\tau_3) + \beta_0\phi_\lambda(\tau_3)
\end{pmatrix}
$$

In general, the above matrix is nonsingular and thus a third variance swap is not redundant in our model since we disconnect the jump intensity from the variance. In the model of Aït-Sahalia, Karaman and Mancini (2015), according to their equations (8) and (9), the corresponding matrix can represented as

$$
\Sigma_1 = \begin{pmatrix}
\phi_v(\tau_1)\sigma_v\sqrt{v_t} & \phi_m(\tau_1)\sigma_m\sqrt{m_t} & \lambda_1[(\mu_j^Q)^2 + \sigma_j^2]\phi_v(\tau_1) \\
\phi_v(\tau_2)\sigma_v\sqrt{v_t} & \phi_m(\tau_2)\sigma_m\sqrt{m_t} & \lambda_1[(\mu_j^Q)^2 + \sigma_j^2]\phi_v(\tau_2) \\
\phi_v(\tau_3)\sigma_v\sqrt{v_t} & \phi_m(\tau_3)\sigma_m\sqrt{m_t} & \lambda_1[(\mu_j^Q)^2 + \sigma_j^2]\phi_v(\tau_3)
\end{pmatrix}
$$

Clearly, the the matrix Σ_1 is singular because its first and third columns are proportional and this, in turn, implies that the third variance swap is redundant. Thus, variance swaps cannot provide independent exposures to two diffusions and one jump, making it difficult to obtain analytic solution to optimal portfolio choice problem involving variance swap. The next result presents the optimal solution for w_{1t}, w_{2t} and w_{3t} in our model.

Proposition 4 *The optimal portfolio weight $w^* = (w_{1t}^*, w_{2t}^*, w_{3t}^*)$ is given by*

$$w^* = \left(\widetilde{\pi}_{b1}^*, \widetilde{\pi}_{b2}^*, \widetilde{\pi}_{q1}^*\right) \Sigma^{-1} \tag{15}$$

where

$$\widetilde{\pi}_{b1}^* = \frac{\gamma_2 \sqrt{v_t}}{\gamma} + \sigma_v \sqrt{v_t} B_1(t), \widetilde{\pi}_{b2}^* = \frac{\gamma_3 \sqrt{m_t}}{\gamma} + \sigma_m \sqrt{m_t} B_2(t),$$

and $\widetilde{\pi}_{q1}^*$ solves the following optimization problem:

$$\sup_{\widetilde{\pi}_{q1} \in [0,\infty)} -\widetilde{\pi}_{q1} E^Q[J^{v,Q}] + \frac{1}{1-\gamma} E^P\left[\left(1 + \widetilde{\pi}_{q1} J^{v,P}\right)^{1-\gamma} e^{\gamma(B_1 + B_3 \beta_0) J^{v,P}} - 1\right]. \tag{16}$$

Proof. See Appendix B. ∎

3.4   Model Performance

Before analyzing the decision of variance swap investments, we first study the performance of four models, including the model investigated by Pan (2002) (termed the "JP" model), the model examined by Egloff, Leippold and Wu (2010) (or the "ELW" model) and the model studied by Aït-Sahalia, Karaman and Mancini (2015) (or the "AKM" model) as well as our model with a self-exciting process for the jump intensity (or the "HJ" model). Both the term structure of variance swap rates and the dynamics of risk premia of risk components over time are then numerically studied. We further investigate the optimal allocations to variance swap contracts in these models, and quantify the losses of economic welfare in investment decisions due to both model mis-specification and parameter mis-specification.

3.4.1 Model Parameters

In order to obtain a comprehensive understanding on the model performance, we use the empirical variance swap rates collected by Aït-Sahalia, Karaman and Mancini (2015), as reported in Table 1. Note that the variance swap rate mean in each maturity category is quoted in volatility percentage units, which is ignored for simplicity unless it is specified. Moreover, all the parameters used for the performance analysis are partially based on the empirical studies conducted by Egloff, Leippold and Wu (2010) and Aït-Sahalia, Karaman and Mancini (2015). To highlight the economic role of jumps in either price or volatility with the framework of the two-factor pure-diffusion model in Egloff, Leippold and Wu (2010), we calibrate all the four models to the empirical term structure of variance swap rates reported in Table 1, and then refer to the ELW model as the benchmark.

		Time to Maturity			
Statistics	2	3	6	12	24
Mean	22.14	22.32	22.87	23.44	23.93
Std	8.18	7.81	7.40	6.88	6.48
Skew	1.53	1.32	1.10	0.80	0.57
Kurt	7.08	6.05	4.97	3.77	2.92

Table 1: **Summary Statistics of Variance Swap Rates**. All the variance swap rates are from Table 1 in Aït-Sahalia, Karaman and Mancini (2015). The sample period is from January 4, 1996 to September 2, 2010. The descriptive statistics, including mean, stand deviation (Std), skewness (Skew) and kurtosis (Kurt), are reported, while the variance swap rate mean in each maturity category is quoted in percentage, and time to maturities are quoted in months.

More specifically, we first obtain the mean term structure of variance swap rates under the objective probability measure P by taking the unconditional expectation as follows:

$$E_{ELW}^P[VS_{t,t+\tau}] = (1 - \phi_v^{ELW}(\tau) - \phi_m^{ELW}(\tau))\theta_m^Q + \phi_v^{ELW}(\tau)\theta_v^P + \phi_m^{ELW}(\tau)\theta_m^P,$$

$$E_{AKM}^P[VS_{t,t+\tau}] = (1 + \lambda_1 E_t^Q[(J^{s,Q})^2])(1 - \phi_v^{AKM}(\tau) - \phi_m^{AKM}(\tau))\theta_m^Q + E_t^Q[(J^{s,Q})^2]\lambda_0$$

$$+ (1 + \lambda_1 E_t^Q[(J^{s,Q})^2])\phi_v^{AKM}(\tau)\theta_v^P + (1 + \lambda_1 E_t^Q[(J^{s,Q})^2])\phi_m^{AKM}(\tau)\theta_{\tilde{m}}^P, \quad (17)$$

$$E_{HJ}^P[VS_{t,t+\tau}] = (1 - \phi_v^{HJ}(\tau) - \phi_m^{HJ}(\tau))\theta_m^Q + \phi_\lambda^0(\tau)\lambda_\infty + \phi_v^{HJ}(\tau)\theta_v^P + \phi_m^{HJ}(\tau)\theta_m^P$$

$$+ \phi_\lambda^{HJ}(\tau)\theta_\lambda^P,$$

in which the mean term structure in the ELW model is a weighted average of the statistical mean of the instantaneous variance rate, θ_v^P, the statistical mean of the central tendency factor, θ_m^P ($\theta_{\tilde{m}}^P$ instead in the AKM model where $\tilde{m} = (\kappa_v^Q m_t + \mu_v^Q \lambda_0)/\tilde{\kappa}_v^Q$ and $\tilde{\kappa}_v^Q = \kappa_v^Q - \mu_v^Q \lambda_1$), and the common risk-neutral (unconditional) long-term mean for both the variance rate v_t and the central tendency m_t, θ_m^Q. In addition to these three factors, the constant jump intensity λ_0 is counted in the AKM model, while both the risk-neutral long-term mean of the jump intensity, λ_∞ and the statistical intensity mean of the jump factor, θ_λ^P are taken into account in the HJ model. In particular, if $\lambda_0 = \lambda_1 = \alpha = \lambda_\infty = \beta_0 \equiv 0$, the jumps in both models vanish, and both models then converge to the ELW model proposed by Egloff, Leippold and Wu (2010) with no jump components. Also, as suggested by Aït-Sahalia, Karaman and Mancini (2015), the JP model can be regarded as a special case of the AKM model when both μ_v^Q and μ_v^P are set as zero (i.e., $\mu_v^Q = \mu_v^P \equiv 0$), but the intensity of jumps employs the same function of variance.

Following the results in Egloff, Leippold and Wu (2010) and Aït-Sahalia, Karaman and Mancini (2015), the analytical formulas of the long-term statistical means, θ_v^P and θ_λ^P are given as follows:

$$
\begin{aligned}
\theta_v^P &= \begin{cases} \frac{\kappa_v^Q \theta_m^P + \lambda_0 \mu_v^P}{\kappa_v^P - \lambda_1 \mu_v^P}, & \text{for ELW, JP and AKM Model,} \\ \frac{\kappa_v^Q \theta_m^P}{\kappa_v^P} + \frac{\alpha \lambda_\infty \mu_v^P}{\kappa_v^P(\alpha - \beta_0 \mu_v^P)}, & \text{for HJ Model,} \end{cases} \\
\theta_\lambda^P &= \frac{\alpha \lambda_\infty}{\alpha - \beta_0 \mu_v^P}, & \text{for HJ Model,}
\end{aligned}
\tag{18}
$$

while $\theta_m^P = \theta_m^Q \kappa_m^Q / \kappa_m^P$ and $\theta_{\tilde{m}}^P = (\kappa_v^Q \theta_m^P + \mu_v^Q \lambda_0)/\tilde{\kappa}_v^Q$.[9]

Moreover, the factor loadings for both the instantaneous variance rate and the central tendency (ϕ_v and ϕ_m) are given in Equation (11). Note that both factor loadings in the AKM model are calculated using a new presentation of $\tilde{\kappa}_v^Q$, and those factor loadings relevant to the jump risk in the HJ model are given in Equation (11). Also, we may obtain all the required weight coefficients of risk components in the JP model by using $\tilde{\kappa}_v^Q$ with $\mu_v^Q = 0$

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[9]Following the similar steps in Section 3.2, these formulas can be obtained by taking limits on the time-t expectations of v and λ under the measure P in these models, respectively. The values of λ_0 and λ_1 in the ELW model are set as zero and μ_v^P is equal to zero in the JP model.

defined in the AKM model.

Panel I: parameters for diffusion components in ELW, JP, AKM and HJ Model									
Parameters	ELW	JP	AKM	HJ	Parameters	ELW	JP	AKM	HJ
κ_v^P	5.060	4.803	5.340	5.340	κ_m^P	0.221	0.234	0.491	0.491
σ_v	0.525	0.419	0.394	0.394	σ_m	0.154	0.141	0.167	0.167
θ_m^P	0.054	0.043	0.038	0.038	γ_2	-1.229	-2.518	-1.964	-2.207
γ_3	-0.704	-0.346	-0.615	-0.239					
Panel II: parameters for jump components in ELW, JP, AKM and HJ Model									
Parameters	ELW	JP	AKM	HJ	Parameters	ELW	JP	AKM	HJ
λ_0	-	4.794	0.176	-	μ_j^Q	-	-0.001	-0.012	-0.012
λ_1	-	114.587	177.715	-	σ_j	-	0.038	0.043	0.043
α	-	-	-	2.472	μ_v^P	-	0	0.001	0.001
λ_∞	-	-	-	5.291	μ_v^Q	-	0	0.002	0.002
β_0	-	-	-	470.276					

Table 2: **Model Parameters**. All the parameters are based on the empirical results in Egloff, Leippold and Wu (2010) and Aït-Sahalia, Karaman and Mancini (2015). All the four models, including the ELW, JP, AKM and HJ model, are re-calibrated to the empirical mean term structure of variance swap rates with five typical maturities (including 2-, 3-, 6-, 12- and 24-month to maturity), as reported in Table 1, and this procedure then results in the RMSEs defined in Equation (19) with 0.0858, 0.0210, 0.0230 and 0.0199, respectively, as reported in .

After specifying the term structure of variance swap rates in all the four models presented in Equation (17), we then conduct the model calibration to the empirical term structure of variance swap rates reported in Table 1 by minimizing the root mean-squared errors (RMSEs) as follows:

$$\text{RMSE}(\Theta; i) = \sqrt{\frac{\sum_{j=1}^N (E_i^P[VS_{t,t+\tau_j}|\Theta] - E_{Market}^P[VS_{t,t+\tau_j}|\Theta])^2}{N}}, \qquad (19)$$

for $i \in \{ELW, JP, AKM, HJ\}$, and Θ denotes the set of model parameters, and $N = 5$ indicates the total number of the time-to-maturities of variance swap contracts in the market. More specifically, we fix those parameters with small standard errors reported in Aït-Sahalia, Karaman and Mancini (2015), and estimate those parameters with large standard errors by calibrating to the empirical mean term structure of variance swap rates reported in Table 1. Following such a procedure, we finally obtain all the required parameters, as reported

in Table 2. It shows that the market prices of both the instantaneous variance and the central tendency factor, γ_2 and γ_3 are negative, despite moderate differences in absolute magnitude. In particular, the highly negative market price of variance risk (γ_2) is confirmed by several studies (see Bakshi and Kapadia (2003), Bondarenko (2004), Carr and Wu (2009) and Todorov (2009) among others).

Also, the negative market prices make the statistical mean-reverting speeds (κ^P) larger and the statistical long-run means (θ_m^P) smaller than their risk-neutral counterparts (κ^Q and θ_m^Q respectively) in all the four models. Linking back to Equation (17), the three long-term means then show the order of $\theta_m^Q > \theta_m^P > \theta_v^P$ in each model, while each of them presents a declining pattern across models duo to the introduction of jumps in variance rate, as suggested in Table 2. Moreover, since the risk neutral mean of the variance jump size is larger than the statistical mean in the AKM and HJ model, i.e., $\mu_v^Q = 0.002 > \mu_v^P = 0.001$, this indicates a negative variance risk premium in variance swap rate defined in Equation (12).

3.4.2 Term Structure of Variance Swap Rates

As suggested in Equation (17), the loading coefficients measure the magnitude of the contemporaneous responses of the variance swap term structure towards unit shocks on risk components (e.g., variance, central tendency and jump). Figure 1 plots the term structure of all risk responses. Among all the four models, the variance risk factor v_t has a transient and dominant contribution (in terms of weight) on the mean term structure of the variance swap rates at short maturities, and such influence gradually declines over maturities, provided that its risk loading coefficient is a monotonically decreasing function of time to maturity. Compared with the other three models, the AKM model puts the highest weights on the volatility factor in the short term. This results from the more contributions made by jumps to capture price variance driven by the large estimator λ_1, and such contributions are eventually reflected by the jump-adjusted weights on the volatility factor due to its specification of the jump intensity that is a function of variance. Accordingly, the similar pattern of the factor loading on v_t can be observed in the JP model.
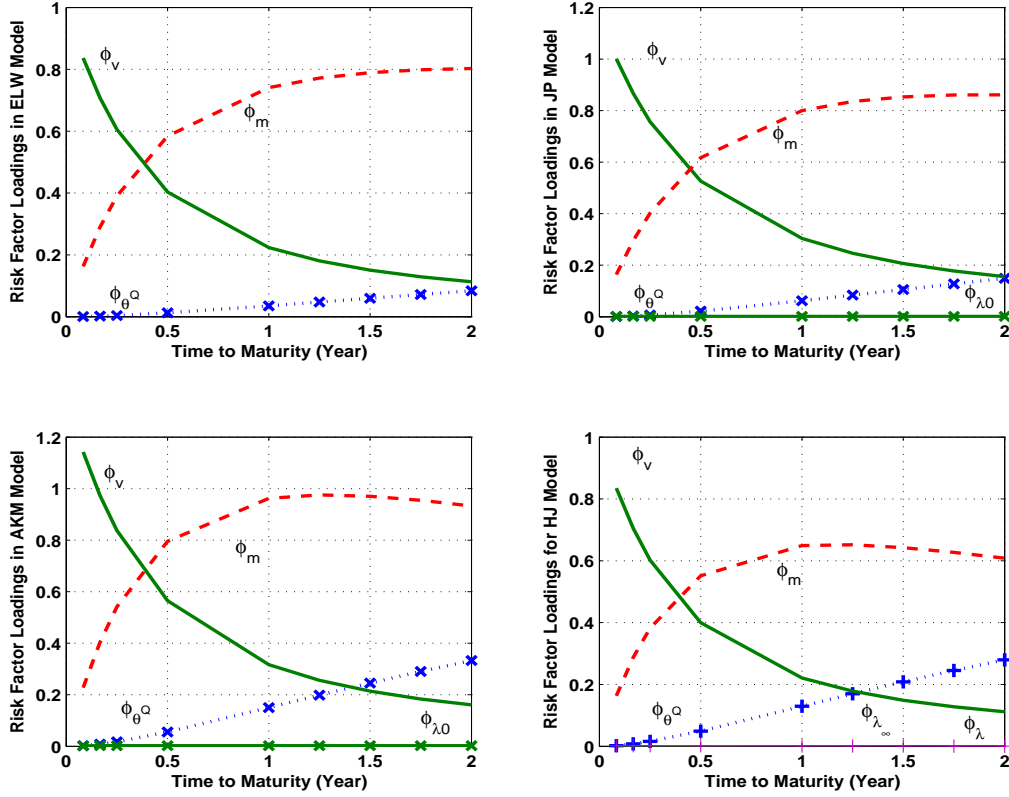
Figure 1: **Factor Loadings of Risk Components.** The contemporaneous response of the variance swap term structure to unit shocks on the instantaneous variance rate v_t (denoted by the solid line), the central tendency factor m_t (denoted by the dashed line) and the jump risk factor λ_t (denoted by the solid line with "+" in the HJ model), while the loading on the long-term mean of the central tendency (θ^Q) is represented by the dotted line with "x". The responses of other risk factors are denoted accordingly (e.g., ϕ_{λ_0} in the JP and AKM model and ϕ_{λ_∞} in the HJ model), but they collapse onto the horizontal axis due to their small values.

In contrast, the impact of the central tendency factor m_t is mainly governed by ϕ_m which is persistent and substantial over time in order to construct an upward-sloping mean term structure. The increasing coefficients of m_t in the ELW and JP model, associated with their relatively large magnitudes, shows that the influence of the central tendency factor intensifies progressively with the increasing of maturity since inception, especially in the presence of jumps in price. Furthermore, if jumps in variance are allowed in both the AKM and HJ model, the coefficients of m_t increase steadily till the medium term and then turn down gradually afterwards, suggesting the declining contemporaneous contributions towards the term structure of variance swap rates. Moreover, the weight of the risk-neutral long-term

mean θ^Q (a constant) monotonically grows as the maturity of variance swap increases in all models as an additional adjustment, which is helpful to mitigate the derivation of variance rates in the long term.

It is distinguishable in both the AKM and HJ model about the manner that the jump risk factor (λ_t) contributes to the responses of the variance swap term structure. Compared to the ELW model, the specification of the jump intensity in the AKM model suggests that the contribution of the jump factor is decomposed into two components: the weight function ϕ_{λ_0} and the weight adjustments on other factors (e.g., v_t, m_t and θ^Q) that result in the substantial upward shifts in the coefficients of v_t, m_t and θ^Q. On the other hand, the self-exciting specification of the jump intensity in the HJ model suggests that the contribution of the jump component is characterized only by two factors: λ_∞ with the weight function ϕ_{λ_∞} and λ_t with ϕ_λ, independent of those risk factors associated with diffusion components. Nevertheless, the values of these two weight functions are relatively small due to the nature of jumps.

Associated with the specifications of the response functions of the risk factors discussed above, we then follow the procedure proposed in Section 3.4.1 to estimate the mean term structure of variance swap rates. As reported in Panel A of Figure 3, the variance swap rates with maturity up to two years fall into a range of $[21\%, 24\%]$ in terms of volatility percentage units.[10] This upward-sloping term structure is certainly consistent with the negative market prices (γ_2 and γ_3) given in Table 2. It seems that the four models are calibrated to the empirical term structure of variance swap rates quite well, and that the HJ model achieves the best calibration with the minimum RMSE of less than 2% (also see Table 2). In light of pricing performance, the JP, AKM and HJ model outperform the ELW model due to very small pricing errors. They even well calibrate to the variance swap rate means at medium and long maturities (e.g., $\tau = 6, 12, 24$ months), while the ELW model always tend to overestimate variance swap rates at short-term and long-term maturities and underestimate them at medium-term maturities. Moreover, those small pricing errors produced by the JP, AKM and HJ model indicate that these three models can achieve the nearly equivalent

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[10]The time-to-maturities include $\tau = 1, 2, 3, 6, 12, 15, 18, 21, 24$ months.

pricing performance for variance swap contracts. More importantly, this further implies that jumps in price or variance or both do play a crucial role in pricing variance swaps.

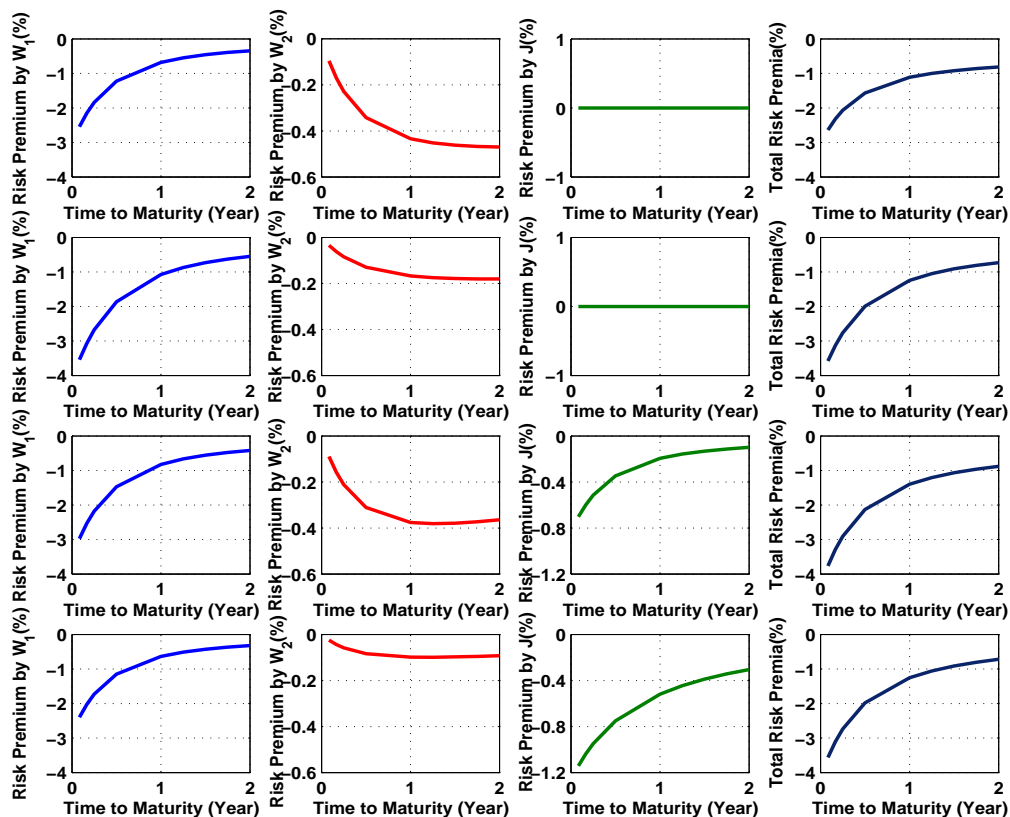### 3.4.3 Term Structure of Variance Swap Risk Premia

Figure 2: **Risk Premium Components in Variance Swap Rates.** Following the left-to-right order in each row, the first three panels present the market prices of three risk factors (e.g., W_1, W_2 and J) in the ELW, JP, AKM and HJ model, respectively, while the fourth panel plots the dynamics of the risk premia that are compensated in total for these risk factors over maturity.

We further investigate the term structure of variance swap risk premia generated by the four models. Since the variance swap rate $VS_{t,t+\tau}$ is a martingale under the measure Q, its associated risk premium that investors would like to pay is then characterized by the drift term of the variance rate dynamics under the measure P, as suggested in Equation (12). In particular, the drift term of the variance rate dynamics in Equation (12) is general in the sense that the ELW, JP and AKM model can be regarded as the special cases by setting $\lambda_t \equiv 0$ for the ELW model and $\beta_0 = 0$ and $\lambda_t = (\lambda_0 + \lambda_1 v_t)$ for the AKM model, while for

the JP model, an extra restriction of $\mu_v^Q = \mu_v^P = 0$ need be imposed, apart from the those requirements in the AKM model. Then the term structure of variance swap risk premia can be obtained by taking the expectation on the variance rate dynamics in Equation (12) under the measure P after compensating for the jump shocks accordingly.

Figure 2 plots the term structure of the risk premium compensated for each individual risk factor, while the price jump premium to a variance swap is equal to zero in the ELW and JP model. More specifically, the upward-sloping term structures of volatility risk premium in all models are presented in the first column, suggesting that risk-averse investors prefer not only to pay a large premium (consistent with the high variance swap rates in Panel A of Figure 3; namely, larger than 21%) but also to take a large expected loss (the relatively high and negative variance risk premium in this column; namely, less than 4%) in short maturities in order to insure against volatility risk. In contrast, the downward-sloping structure of risk premium against stochastic central tendency is reported in the second column, indicating that investors would like to pay an extra premium, despite of its relatively smaller absolute magnitude (less than 0.4%), in order to mitigate uncertainty in central tendency as maturity increases. In addition to the risk premia compensated for variance risk and central tendency, a negative and substantial premium (less than 1.2%) need be counted for variance jump risk, as reported in the third column, which further leads to a downward shift in the term structure of variance swap risk premia. As a result, the sum of the risk premia compensated for three risk factors reported in the first three columns is sequentially reported in the final column.

Panel B in Figure 3 represents the term structure of risk prima plotted in the final column in Figure 2 with more details. It is clearly that less risk premium need be paid by investors in the ELW model, particularly in short maturities, due to no premium for jump risk, which could make a variance swap more affordable. After introducing jumps in price and variance, an extra premium is required to compensate jump risk, for example, in the JP, AKM and HJ model. On the one hand, the specification of the jump intensity that is a function of instantaneous variance in the JP and AKM model implies that high volatility in short maturities drives up demands for high volatility risk compensations in short term,

as suggested in Figure 2. On the other hand, this specification further suggests that due to jumps in variance in the AKM model the premium for the central tendency risk factor also rises as maturity increases in order to mitigate uncertainty in central tendency.[11] This then makes a variance swap more expensive. Consequently, the total risk premiums in the JP and HJ model are higher than those in the ELW model, but lower than those in the AKM model, despite their very close pricing performance. Interestingly, the term structure of variance swap risk premia in both models nearly coincides with each other with mean -0.0076 and standard deviation 1.28%.

3.5 Numerical Analysis of Variance Swap Investments

The preceding analysis suggests that the JP, AKM and HJ model can be calibrated to the empirical mean term structure of variance swap rates quite well, but less risk premia are required in the HJ model, as suggested in Figure 3. This then indicates that the JP and HJ model can be regarded as good alternatives for the AKM in terms of valuing variance swap rate contracts. However, it is unclear how the absence of jumps in variance may affect investor's decisions on variance swap investment. Since the AKM model does not provide an analytical solution to the problem of variance swap allocation, we mainly consider the other three models (namely, the ELW, JP and HJ model) in this section.

Following the theoretical results on optimal allocations in Section 3.3, and the parameters on the variance swap rate dynamics estimated in Section 3.4.1, we analyze the optimal allocations to variance swap contracts in the absence of stock index investment, study the role of jumps in the construction of of variance swap portfolios, and further exam the cost of economic welfare in investing these contracts due to model mis-specification.

3.5.1 Optimal Variance Swap Allocations

Unlike the stream of literature that attempts to rationalize the magnitude of the risk premium based on various economic issues, we instead study how a trader allocates her wealth to

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[11] The stochastic variations in variance is captured by a jump factor in price indirectly (in the JP model) or in variance directly (in the HJ model), which is independent of the central tendency factor.

variance swap contracts in order to benefit from the risk premium dynamics and further investigate the impact of jumps in variance on her asset allocations.[12] We now assume that the trader prefers to allocate the initial wealth W_t at time t between the money market and the variance swap market. The trader has access to the money market account to balance out the investments by earning a risk-free interest rate. The variance swap contracts are initialized with zero costs and so they have zero initial values.

In the absence of jumps in variance, as discussed before, the variance swap market is complete, and its dynamics are driven by a two-factor variance risk structure so that the values of the variance swap rates across all maturities are determined by two sources of variations. Accordingly, the trader just need choose any two variance swap contracts with distinct time to maturities (say $0 < \tau_1 < \tau_2 < \infty$ without loss of generality), which could sufficiently span all the sources of risks in the variance swap market. When the investor only invests in the money market and variance swap contracts, the optimal portfolio weight of her wealth invested in these two contracts, $w^* = (w_{1t}^*, w_{2t}^*)$ is equal to $w^* = (\tilde{\pi}_{b1}^*, \tilde{\pi}_{b2}^*)\Sigma^{-1}$, as given in Equation (15) with $\tilde{\pi}_{q1}^* \equiv 0$ (and $B_3 \equiv 0$ in Proposition 3 as well), while the 2×2 matrix Σ is given as follows:

$$
\Sigma = \begin{pmatrix} \phi_v(\tau_1)\sigma_v\sqrt{v_t} & \phi_m(\tau_1)\sigma_m\sqrt{m_t} \\ \phi_v(\tau_2)\sigma_v\sqrt{v_t} & \phi_m(\tau_2)\sigma_m\sqrt{m_t} \end{pmatrix}
$$

and ϕ_v and ϕ_m are given in Equation (11), which is equivalent to the formulas in Egloff, Leippold and Wu (2010)(see Equation (43) and (44) in page 1298). It is clear that these results are valid in both the ELW and JP model but with distinct parameter sets that are reported in Table 2. Note that for the JP model, the exposures of the two contracts to the sources of risk, ϕ_v and ϕ_m in the matrix Σ are obtained by setting $\mu_v^Q = \mu_v^P = 0$ in the AKM model.

Inspecting the optimal investment decisions in the variance swap contracts under the

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[12]For these issues, see Bates (2008) for path-dependent preferences, Liu, Pan and Wang (2005) and Jin and Zhang (2012) for ambiguity aversion, Bollen and Whaley (2004) and Gârleanu, Pedersen and Poteshman (2009) for net-buying and demand pressure, Isaenko (2007) for short-selling constraints and Santa-Clara and Saretto (2009) for margin requirements among others.

two-factor variance risk specification (e.g., the ELW and JP model), Egloff, Leippold and Wu (2010) suggest that the optimal allocations in the two variance swap contracts at short investment horizons depend not only on the market prices of both the variance risk (γ_2) and the central tendency risk (γ_3), but also on the exposures of the two contracts towards the risk factors, ϕ_v and ϕ_m, as suggested by the matrix Σ.

More specifically, investment in the short-term contract is more sensitive to the market price of the variance risk, while investment in the long-term contract depends more on the market price of the central tendency risk, owing to the distinct patterns of risk loadings of these two factors over time. This further implies that the position in the short-term contract with maturity τ_1 is mainly determined by the variance risk price γ_2 embedded in this contract through the diffusion exposure $\tilde{\pi}_{b1}^*$, especially when the maturity of the long-term contract is sufficiently long, $\tau_2 \to \infty$, leading to the dominating loading on the central tendency factor ($\phi_v(\tau_2) \ll \phi_m(\tau_2)$). In contrast, the position in the long-term contract is mainly determined by the market price of the central tendency risk (γ_3) through the diffusion exposure $\tilde{\pi}_{b1}^*$, if the short-term contract has an extremely short maturity ($\tau_1 \to 0$), which results in the dominant risk loading on the variance risk factor ($\phi_v(\tau_1) \gg \phi_m(\tau_1)$), as suggested in Figure 1. This can be seen clearly from the portfolio weights $w^* = (\tilde{\pi}_{b1}^*, \tilde{\pi}_{b2}^*)\Sigma^{-1}$. Considering the example where $\tau_1 = 2$ months and $\tau_2 = 2$ years and the ELW model's parameters are given in Table 1, the sensitive matrix Σ^{-1} is given by

$$\Sigma^{-1} = \begin{pmatrix} 13.1570 & -4.7567 \\ -5.9110 & 36.9530 \end{pmatrix}.$$

And thus,

$$w^* = (13.1570\tilde{\pi}_{b1}^* - 5.9110\tilde{\pi}_{b2}^*, -4.7567\tilde{\pi}_{b1}^* + 36.9530\tilde{\pi}_{b2}^*).$$

In this case, the investor uses short-term and long-term variance swaps to exploit the risk premia γ_2 through $\tilde{\pi}_{b1}^*$ and γ_3 through $\tilde{\pi}_{b2}^*$, separately, as indicated by the positive numbers 13.1570 and 36.9530. In the meantime, the investor takes positive exposure to the second Brownian motion in the short-term contract (-5.9110) to offset the negative exposure of

the long-term contract to the second Brownian motion. For the same reason, the investor takes positive exposure to the first Brownian motion in the long-term contract (-4.7567) to offset the negative exposure of the short-term contract to the first Brownian motion. In fact, negative market prices of the two risk factors result in short positions -0.5271 and -0.9634 in both contracts. Moreover, the trading positions in the two variance swap contracts may dependent on the relative magnitude of the market prices of the two sources of risk, γ_2 and γ_3, as their maturity gap tends to be relatively moderate. Then the optimal allocations could involve short positions in the short-term variance swap contract, but long positions in the long-term variance swap contract, if the variance price term γ_2 is extremely larger than the central tendency price term γ_3 in absolute value.

Similar to the empirical results in Egloff, Leippold and Wu (2010), Figure 4 shows the dependence of the optimal variance swap investments on the market prices of the risk factors: the variance risk and the central tendency risk.[13] In the ELW model, the two-factor variance market suggests that any two variance swap contracts can span all the risk sources. As we discussed before, Panel A and B show that the optimal investments in the variance swap contracts involve short positions in the short-term contract and long positions in the long-term contract when the distance between the two maturities is moderate. Also, Figure 4 suggests that the distance between the two surfaces widens as the difference between the market prices of risk factors increases, while the smaller maturity difference between the two contracts may lead to the more holdings of each contract accordingly. Furthermore, the introduction of jumps in stock prices in the JP model does not cause substantial impact on the allocations decisions to the variance swap contracts, as the variance market is still completed and can be spanned by the combinations of any two contracts. Although this manner may result in the upward shifts in the risk loadings on both the volatility factor and the central tendency factor, the optimal investment is still determined within the ELW model. Therefore, jumps in stock prices lead to limited effects on the optimal investment in

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[13]When the market prices of the risk factors vary, unlike the way in Egloff, Leippold and Wu (2010), we adjust all the estimators under the measure P by fixing the estimators in the measure Q in order to reflect their impact on the intertemporal hedging demand.

variance swap contracts, apart from the minor differences in magnitude.[14]

Furthermore, the presence of jumps in variance then makes the variance swap market incomplete. Under this circumstance, the dynamics of this market are now driven by a three-factor variance risk structure due to the three sources of variations. As a result, another variance swap contract with maturity τ_3 is required to span all the sources of variations in this three-factor model, which leads to the optimal investment as notional in the three variance swap contracts in terms of the fractions of wealth given in Equation (11). Note that the choice of variance swap contracts is actually independent from specific maturities, yet only dependent on the life of each contract, no matter whether the variance swap market is complete or not.

Figure 5 plots the allocations to variance swap contracts in the presence of jumps in variance for a 2-month investment horizon. The inclusion of the third contracts further completes the variance market. However, due to the small estimators for jump sizes in variance in both the measure P and Q, as reported in Table 2, these values then cause a linearity problem to the 3×3 matrix, Σ in Equation (15). This then makes its inverse matrix large and in turn the holdings of each contract are very large as well, as shown in Figure 5. Similar to the results produced in the ELW model in Figure 4, the trading positions in the three variance swap contracts depend on the relative magnitudes of the market prices of the two sources of risk, γ_2 and γ_3. Also, when the third contract with maturity τ_3 is included in the portfolio, the smaller maturity gap between the two contracts with maturity τ_1 and τ_2 may still lead to more holdings on these contracts. Even when the maturity gap between the two contracts is moderate, the positioning of the third contract may cause substantial increases in the holdings of these three contracts, for example, as suggested in Panel C and D of Figure 5.

More importantly, Figure 5 suggests that the optimal investment decision is irrelevant to the position of the third contract with maturity τ_3 in spite of differences in trading volume. In the presence of jumps in variance, it is always optimal to take long positions in medium-

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[14]We have examined the impact of jumps on variance swap investments, and found that the differences in the holdings of either the short-term contracts or the long-term contracts are less than 7 in absolute magnitude value. As a result, the optimal investments in variance swap contracts are not reported here.

term variance swap contracts and short positions in both short-term and long-term contracts. Also, the trader should sell even more long-term contracts than short-term ones. For a better understanding, we rewrite the portfolio weights $w^* = (\tilde{\pi}_{b1}^*, \tilde{\pi}_{b2}^*, \tilde{\pi}_{q1}^*)\Sigma^{-1}$ in the example where $\tau_1 = 2$ months, $\tau_2 = 1$ year and $\tau_3 = 2$ years, the sensitive matrix Σ^{-1} is given by

$$\Sigma^{-1} = \begin{pmatrix} 77.4910 & -359.3994 & 346.9892 \\ 13.8305 & -114.1198 & 197.7839 \\ -1.9143 & 12.2678 & -12.1893 \end{pmatrix}.$$

And thus,

$$w_1^* = 77.4910\tilde{\pi}_{b1}^* + 13.8305\tilde{\pi}_{b2}^* - 1.9143\tilde{\pi}_{q1}^*,$$

$$w_2^* = -359.3994\tilde{\pi}_{b1}^* - 114.1198\tilde{\pi}_{b2}^* + 12.2678\tilde{\pi}_{q1}^*,$$

$$w_3^* = 346.9892\tilde{\pi}_{b1}^* + 197.7839\tilde{\pi}_{b2}^* - 12.1893\tilde{\pi}_{q1}^*.$$

We can make the following observations from the above optimal portfolio weights (e.g., $w_1^* = -6.6042, w_2^* = 31.4253$ and $w_3^* = -30.9234$). First, for each variance swap, the sensitivity to jump is much smaller in magnitude compared with the sensitivities to the two diffusion risks. The reason for this is that the value of ϕ_λ is very small due to small jump size in variance. Second, for each variance swap, the sensitivity to the variance exposure is much larger than those to other two exposures in magnitude. Third, the sensitivities to all the risk factors in the long-term contract are much larger than those in the short-term one in magnitude. Given the term structure of variance swap risk premia in Panel B of Figure 3, the trader may gain from the unexpected high frequency jumps in variance (governed by $\beta_0 = 470.2764$) such that the premiums for medium-term contracts are paid out by the compensations from short positions in both short-term and long-term contracts.[15]

Due to the small market price compensated for jump risk, as reported in Table 2, we then adjust μ_v^Q by fixing μ_v^P in order to reflect the sensitivity of optimal investments in

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[15]This then implies that the trader may follow a reverse strategy to maximize her expected utility if the frequency of jumps in variance is low (governed by a relatively small β_0): taking short positions in medium-term variance swap contracts and long positions in both short-term and long-term contracts.

variance swap contracts towards jump risk, i.e., $\mu_v^Q = \mu_v^P - \gamma_J$ where γ_J is chosen to ensure the positivity of the long-term jump intensity λ_t, e.g., $\mu_v^Q \leq \alpha/\beta_0$. Similar to the results in Figure 5, Figure 6 suggests that it is still optimal to take long positions in medium-term contracts, and short positions in both short-term contracts and long-term contracts. Interestingly, the optimal investments are insensitive to small changes in the market price of jump risk when it is relatively high, and then turns to be very sensitive when the market price of jump risk increases up to 0.4%. In particular, when jump risk in variance is compensated by a low market price, the relatively flat surfaces of the holdings in each contract suggest that jump risk has minor impact on the allocation to variance swap contracts, and the trader's investment decision is mainly affected by variance risk and central tendency risk. However, jump risk plays a substantial role in investment decision when its mark price turns to be high. In order to maximize her expected utility in the investment horizon $T = 2$ months, the trader then exploits opportunities by rapidly increasing holdings of both medium- and long-term contracts, as demonstrated in Figure 6.

3.5.2 Optimal Hedging Demands

Proposition 4 shows that the optimal allocation to the variance swap contracts consists of a myopic component that is the optimal portfolio with a constant opportunity set, and an intertemporal hedging demand that a trader may ask for to reduce the impact of shocks to the indirect utility of wealth when facing stochastic opportunities. As a result, the portfolio rule $w^* = (w_{1t}^*, w_{2t}^*, w_{3t}^*)$ is the sum of the myopic demand and the intertemporal hedging demand. In this context, the stochastic variance risk (including the central tendency risk) and jump risk represents those stochastic investment opportunities, which induces induces an intertemporal hedging demand when we invest in the variance swap contracts alone.

In the literature, the hedging demand for volatility is not significant in a realistic stock-bond portfolio problem within a stochastic variance environment, as discussed by Buraschi, Porchia and Trojani (2010). It is necessary to investigate whether this empirical observation still holds in the context of variance swap investments. For this reason, we may set $\sum^{-1} = (\tilde{\sigma}_{i,j})$ for each variance swap contract j $(j = 1, 2, 3)$ based on Proposition 4, and further

obtain the following demands on this contract at time t:

$$\text{Total myopic demand:} M = \frac{1}{\gamma}(\tilde{\sigma}_{1,j}\gamma_2\sqrt{v_t} + \tilde{\sigma}_{2,j}\gamma_3\sqrt{m_t});$$

$$\text{Hedging demand for } v_t : H_v = \tilde{\sigma}_{1,j}\sigma_v\sqrt{v_t}B_1(t); \qquad (20)$$

$$\text{Hedging demand for } m_t : H_m = \tilde{\sigma}_{2,j}\sigma_m\sqrt{m_t}B_2(t).$$

By virtue of these demands, we may work out the hedging ratios for both the volatility risk v and the central tendency risk m, reported as the percentages of the myopic portfolio in Table 3 separately. These ratios clearly show that the intertemporal hedging demands, for example, for volatility risk and central tendency risk, are indeed significant in the context of variance swap investments. In particular, the hedging demands for volatility vary from 0.040 to 0.088 when the degree of risk aversion increases from $\gamma = 2$ to $\gamma = 40$ within the investment horizon of $T = 20$ years, while the changes in the hedging demands for the central tendency risk can be more substantial, ranging from 0.005 to 0.228 when the trader tends to be more risk averse, indicated by the increasing magnitude from $\gamma = 2$ to $\gamma = 40$.

Figure 7 plots the total hedging demands for the volatility risk v_t and the central tendency risk m_t in terms of the percentages of the myopic portfolio by varying investment horizon (in the left panel) and relative risk aversion (in the right panel). More specifically, the sensitivity of the total hedging demands to the investment horizon (T) is consistent to the empirical results in Buraschi, Porchia and Trojani (2010) in magnitude (e.g., approaching 25% for $T = 20$ years), but the total hedging demands are more sensitive to the degree of relative risk aversion (γ), e.g., close to 30% for $\gamma = 40$, compared with the empirical results in Buraschi, Porchia and Trojani (2010) (close to 18% for $\gamma = 40$). This may be partially caused by the unique feature of volatility trading, as in nature the investors are usually risk averse to volatility.[16]

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[16]We also re-exam the sensitivities of the results in Table 3 and Figure 7 by varying the parameter estimators in Table 2 and find these results are robust regarding the changes in parameters.

3.5.3 Economic Welfare in Variance Swap Investments

Previously, we analyze the optimal allocations to the variance swap contracts when the variance market is driven by the 2- or 3- factor structural models, respectively. It is interesting to further investigate the economic costs for the trader who invests heavily in this market if the dynamics of the variance swap is mis-specified by a 2-factor model (e.g., the ELW model or the JP model), while the HJ model is the true specification for its dynamics.

We follow the literature to evaluate the economic costs by a certainty equivalent loss (CE) defined by: The utility cost, CE, of following the suboptimal strategy $w = (n_{1t}, n_{2t})^\top$ satisfies the equation below:

$$J(t, W_t(1 - CE), X_t) = J^{(1)}(t, W_t, X_t),$$

where $J(t, W_t, X_t)$ is the indirect value of the portfolio choice problem in HJ model given by Proposition 4 and $J^{(1)}(t, W_t, X_t)$ is the value function provided in Proposition below corresponding to the the suboptimal strategy $w = (n_{1t}, n_{2t})^\top$ in HJ model. Intuitively, CE is the percentage of initial wealth an investor is willing to pay to switch from the suboptimal strategy w to the optimal strategy w^*. The following result presents the calculation of CE.

Proposition 5 *Under the above assumptions, we have the following result:*

$$J^{(1)}(t, W_t, X_t) = \frac{W_t^{1-\gamma}}{1-\gamma}[f(t, X_t)]^\gamma = \frac{W^{1-\gamma}}{1-\gamma}\left[e^{A^{(1)}(t)+B_1^{(1)}(t)v_t+B_2^{(1)}(t)m_t+B_3^{(1)}(t)\lambda_t}\right]^\gamma \qquad (21)$$

where the functions $A^{(1)}(t)$, $B^{(1)}(t) = (B_1^{(1)}(t), B_2^{(1)}(t))^\top$ and $B_3^{(1)}(t)$ satisfy the following equations:

$$\frac{dA^{(1)}}{dt} + \kappa_m^P\theta_m^P B_2^{(1)} + \alpha\lambda_\infty B_3^{(1)} + \frac{1-\gamma}{\gamma}r = 0,$$

$$\frac{dB_1^{(1)}}{dt} - [\kappa_v^P - (1-\gamma)\psi_v\sigma_v^2]B_1^{(1)} + \frac{1}{2}\gamma\sigma_v^2\left(B_1^{(1)}\right)^2 + \frac{1-\gamma}{\gamma}\sigma_v\gamma_2\psi_v + \frac{\gamma-1}{2}\psi_v^2\sigma_v^2 = 0,$$

$$\frac{dB_2^{(1)}}{dt} + \kappa_v^Q B_1^{(1)} - [\kappa_m^P - (1-\gamma)\psi_m\sigma_m^2]B_2^{(1)} + \frac{1}{2}\gamma\sigma_m^2\left(B_2^{(1)}\right)^2 + \frac{1-\gamma}{\gamma}\sigma_m\gamma_3\psi_m + \frac{\gamma-1}{2}\psi_m^2\sigma_m^2 = 0,$$

$$\frac{dB_3^{(1)}}{dt} - \alpha B_3^{(1)} + \frac{\gamma-1}{\gamma}\widetilde{\pi}_{q1}\mu_v^Q + \frac{1}{\gamma}E^P\left[\left(\widetilde{\pi}_{q1}J^{v,P} + 1\right)^{1-\gamma}e^{\gamma(B_1^{(1)}+B_3^{(1)}\beta_0)J^{v,P}} - 1\right] = 0,$$

with $A^{(1)}(T) = B_1^{(1)}(T) = B_2^{(1)}(T) = B_3^{(1)}(T) = 0$. *Here* $\psi_v = n_{1t}\phi_v(\tau_1) + n_{2t}\phi_v(\tau_2)$ *and* $\psi_m = n_{1t}\phi_m(\tau_1) + n_{2t}\phi_m(\tau_2)$. *The utility cost, CE, of following the suboptimal strategy* $w = (n_{1t}, n_{2t})^\top$ *satisfies the equation below:*

$$J(t, W_t(1 - CE), X_t) = J^{(1)}(t, W_t, X_t),$$

and thus

$$CE = 1 - \left[e^{A^{(1)}(t) - A(t) + (B_1^{(1)}(t) - B_1(t))v_t + (B_2^{(1)}(t) - B_2(t))m_t + (B_3^{(1)}(t) - B_3(t))\lambda_t} \right]^{\frac{\gamma}{1-\gamma}}$$

Proof. See Appendix C. ∎

Unlike the indirect value function $J(t, W_t, X_t)$, the above result suggests that the value function $J^{(1)}(t, W_t, X_t)$ depends on the maturities of the two variance swaps through the functions $A^{(1)}(t), B_1^{(1)}(t), B_2^{(1)}(t)$ and $B_3^{(1)}(t)$. The economic cost thus depends on the two maturities. Also, Proposition 5 suggests that the jump risk exposure $\tilde{\pi}_{q1}$ play a crucial role in determining the magnitude of CE. It is then necessary to evaluate $\tilde{\pi}_{q1}$ before numerically assessing the economic cost of following a suboptimal strategy w.

As well-understood in the literature of portfolio choice problem in jump-diffusion model, see, e.g., Proposition 1 of Liu, Longstaff and Pan (2003), the investor must restrict her jump exposure $\tilde{\pi}_{q1}$ to guarantee that her wealth remains positive when jump occurs. In particular, $\tilde{\pi}_{q1}$ satisfies: $\tilde{\pi}_{q1} \geq 0$ since the support of the variance jump size J_v^P is $[0, \infty)$. As a result, before presenting the empirical results for CE given by Proposition 5, it is interesting to examine whether or not the restriction on π_{q1} is violated by the suboptimal strategy w. The reason for this is that, as mentioned in Egloff, Leippold and Wu (2010), an investor often takes an extreme short position in a variance swap due to significantly negative variance risk premium in variance swap rate, and thus this is very likely to lead violation of the restriction. Table 4 shows that all the jump exposures $\tilde{\pi}_{q1}$s caused by the ELW model are negative across the various investment horizons, suggesting that the trader are subject to the substantial

jump risk exposure by ignoring jumps in volatility.[17] This table then confirms our concern, that is, the restriction is overwhelmingly violated by the the suboptimal strategy w, implying 100 percent loss.

To further evaluate the economic costs in the case where the suboptimal strategy w is feasible in HJ model, we change the support $[0, \infty)$ of the variance jump size J_v^P into $[0, 1]$. This means that the variance can jump at most 100 percent, and then the restriction on $\tilde{\pi}_{q1}^*$ in the optimization problem in Proposition 4 becomes $\tilde{\pi}_{q1}^* > -1$.[18] In the meantime, this change does not affect the variance swap rate due to the negligible probability $P(J_v^P > 1)$. As a result, Table 5 report the economic costs of switching from a suboptimal strategy w generated in ELW model or JP model to the optimal strategy w^* generated in HJ model.

Under a two-factor variance structure, two variance swap contracts would be sufficient to span the market. As shown in Figure 4, two typical combinations of contracts, one with the 2-month and 2-year contracts ($\tau_1 = 2/12$ and $\tau_2 = 2$) and the other with the 6-month and 1-year contracts ($\tau_1 = 6/12$ and $\tau_2 = 1$), are used to quantify the economic costs that the trader has to bear due to model mis-specification. The first panel in Table 5 reports the economic costs for the trader with $\gamma = 5$ with the various investment horizons (T) by assuming the rolling-over of the specific variance swap contracts). More specifically, the economic costs in two combinations of contracts are very close and steadily increasing with the growth of T, suggesting that the economic costs in both ELW model and JP model are less dependent on the maturity gap, but sensitive to the length of investment horizon. Compared with the JP model, the ignorance of jumps in both price returns and variance can make the ELW model produce relatively higher economic costs when the investment horizon increases from 6 months to 30 years, implying that incorporating jumps into price returns is of the first-order importance for variance swap investments.

The second panel in Table 5 further reports the economic costs for the trader with $\gamma = 40$.

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[17]The jump exposures caused by the JP model are still negative with the larger absolute magnitude, and thus they are not reported. Also, we exam the jump exposure $\tilde{\pi}_{q1}$ of both the ELW and JP model if the AKM model is assumed to be the true model, and find that all the jump exposures are negative but with the relatively smaller absolute values, implying the underestimation of the unhedged jump risk.

[18]This assumption is supported by the dynamics of the CBOE Volatility Index (or VIX) over the past ten years at http://www.cboe.com/delayedquote/advchart.aspx?ticker=VIX.

Similar to the case of $\gamma = 5$, the economic costs of the contract combinations in both models are still increasing significantly with the length of the investment horizon (T). Also, the magnitudes of economic costs shift down overwhelmingly across all investment horizons when the degree of risk aversion turns to be much higher from $\gamma = 5$ to $\gamma = 40$. In particular, the economic cost of each contract combination in JP model is persistently reduced for each investment horizon (T), compared with those costs reported in the first panel. This seems to suggest that for the extremely risk-averse trader, the incorporation of jumps into price returns has little impact on economic costs in the context of asset allocation to variance swap contracts, compared with those produced by the ELW model with the moderate risk aversion (e.g., $\gamma = 5$). This is caused partially because the extremely high risk aversion ($\gamma = 40$) enforces the trader to make much smaller investments in variance swaps and thus the impact of jump components in model mis-specification is mitigated to a large extent.

Table 5 has demonstrated the impact of jumps in prices on economic costs caused by model mis-specification.[19] This suggests that it is interesting to further investigate the impact of jumps on variance on economic costs, given the small estimators of μ_v^P and μ_v^Q in Table 2 that capture jump size in variance under the measure P and Q. Table 6 reports the economic costs caused by ELW model with a range of jump size in variance for a portfolio of the 2-month and 2-year variance swap contracts (i.e., $\tau_1 = 2/12, \tau_2 = 2$).[20] It clearly shows that apart from the investment horizon (T), jumps in variance do have substantial impact on economic costs when the dynamics of variance is improperly specified. That is, large jumps in variance can result in high economic costs, which again emphasizes the importance of incorporating jumps in variance in the context of variance swap investments.

We further investigate the sensitivity of economic costs towards parameter mis-specification in HJ model. Suppose that one parameter in Table 2 is mis-specified. We then use Proposition 3 and 4 to obtain the optimal portfolio wight $w = (n_{1t}, n_{2t}, n_{3t})$. Then, the corresponding

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[19]Given the increasing value of μ_v^P, we adjust the value of $J^{v,P}$ accordingly by scaling up a constant μ_v^P/m_0 with $m_0 = \mu_v^P - exp(-1/\mu_v^P)(1 + \mu_v^P)$ to ensure $E[J^{v,P}\mu_v^P/m_0] = \mu_v^P$ for $J^{v,P} \in [0, 1]$.

[20]We also exam the economic costs caused by ELW model and JP model for a portfolio of the 6-month and 1-year variance swap contracts (i.e., $\tau_1 = 6/12, \tau_2 = 1$), and obtain the consistent results with those reported in Table 5 and 6.

CE is still calculated by Proposition 5 with the following modifications:

$$\psi_v = n_{1t}\phi_v(\tau_1) + n_{2t}\phi_v(\tau_2) + n_{3t}\phi_v(\tau_3),$$

$$\psi_m = n_{1t}\phi_m(\tau_1) + n_{2t}\phi_m(\tau_2) + n_{3t}\phi_m(\tau_3), \tag{22}$$

$$\tilde{\pi}_{q1} = \sum_{i=1}^{3} n_{it}(\phi_v(\tau_i) + \beta_0\phi_\lambda(\tau_i)).$$

In the literature, the CE of parameter mis-specification is relatively smaller than the model mis-specification. But in variance swap investments, it may be different because π_{q1} can be easily become negative for incorrect parameters, as implied by the preceding analysis.[21] Table 7 confirms our concern. It clearly shows that the trader may be easily bankrupt when the parameter κ_v^P is underestimated, and may suffer economic costs when κ_v^P is overestimated. Furthermore, Table 8 suggests that the trader certainly suffers relatively small economic costs, no matter whether the parameter κ_m^P is underestimated or overestimated. In both case, the economic costs tend to be large when the investment horizon T increases from $T = 6$ months to $T = 20$ years for a relative low degree of risk-aversion ($\gamma = 2$). Recall in the previous section that the hedging ratios for volatility are significant in the context of variance swap investments. These results then suggests that the proper estimation strategy of parameters, for example, μ_v^P and μ_m^P among others that capture the dynamics of volatility, may significantly mitigate the impact of parameter mis-specification so that the trader may suffer less economic costs when entering into the volatility market.

4 Dynamic asset allocation for stocks, bonds and cash

As a second application of the theoretical results developed in Section 2, we now examine how jumps in stock return affects the optimal cash-bond-stock mix in a dynamic asset allocation model where an investor can trade one stock, two bonds and cash. A closely related problem is the asset allocation puzzle raised in Canner et al. (1997). Namely, the empirical evidence

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[21]We mainly report the economic costs by mis-specifying μ_v^P and μ_m^P given their relative large magnitudes after varying the estimators of the parameters reported in Table 2. To conduct the analysis, we restrict $\tilde{\pi}_{q1}^* \geq 0$ in the optimization problem in Proposition 4.

documented in their paper shows that strategic asset allocation advices tend to recommend a higher bond/stock ratio for a more risk averse investor. This finding, however, is inconsistent with the Tobin (1958) Separation Theorem that the ratio of bonds to stocks in the optimal portfolio is the same for all investors regardless of the investor's risk aversion.

Bajeux-Besnainou et al. (2001) and Brennan and Xia (2000) relate the puzzle to a hedging component to a stochastic interest rate and provide an elegant solution to the asset allocation puzzle. More specifically, as pointed by Lioui (2007), the puzzle can be resolved under the assumption that one or several bonds can perfectly hedge the risk from the interest rate and the market price of risk. This approach is extended by Lioui (2007), which demonstrates that the puzzle may be still a puzzle for a bond market and stochastic market prices of risk if the hedging assumption is invalid. All of these studies assume that short term interest rate and stock return follow pure-diffusion processes. Our framework generalizes these studies by incorporating jumps in stock returns and examining the roles of risk aversion in determining the optimal cash-bond-stock mix. In particular, we will show that unlike the pure-diffusion model in Lioui (2007), there is no clear-cut answer to the bond/stock ratio puzzle in a jump-diffusion model despite the aforementioned hedging assumption. This finding strengthens the claim made by Lioui (2007) that the asset allocation puzzle is still a puzzle.

Like Lioui (2007), we adopt a two factor Vasicek (1977) term structure model which is a simplified version of the multi-factor models in Sanjvinatsos and Wachter (2005). We extend it by adding a jump component in the stock price. The model assumes the following dynamics under the physical measure P:

$$r(X(t), t) = \delta_0 + \delta^\top X(t), \tag{23}$$

$$dX(t) = K(\theta - X(t))dt + \sigma_X dZ(t), \tag{24}$$

$$\bar{\Lambda}(t) = \bar{\lambda}_1 + \bar{\lambda}_2 X(t), \tag{25}$$

$$\frac{d\phi(t)}{\phi(t)} = -r(t)dt - \bar{\Lambda}(t)^\top dZ(t) \tag{26}$$

where $r(t)$ is the short term interest rate, $X(t)$ is a 2×1 vector of state variables, $\bar{\Lambda}(t)$ is a price of risk, and $\phi(t)$ is a pricing kernel, $Z(t) = (Z_1(t), Z_2(t))^\top$ is a standard 2-dimensional

Brownian motion. $\delta_0 \in \mathcal{R}, \delta \in \mathcal{R}^{2\times 1}, K \in \mathcal{R}^{2\times 2}, \theta \in \mathcal{R}^{2\times 1}, \sigma_X = (\sigma_{X_{ij}})_{1 \leq i,j \leq 2}$ is a 2×2 non-singular matrix, $\bar{\lambda}_1 \in \mathcal{R}^{2\times 1}$, $\bar{\lambda}_2 \in \mathcal{R}^{2\times 2}$, all of these parameters are assumed to be constants.

As indicated by Sanjvinatsos and Wachter (2005), the nominal bond price shall follow

$$\frac{dP_i(t)}{P_i(t)} = (A_2(\tau_i)\sigma_X\bar{\Lambda}(t) + r(t))dt + A_2(\tau_i)\sigma_X dZ(t), i = 1, 2, \tag{27}$$

where $\tau_i = T_i - t$ and T_i denotes the maturity date of bond i, $\tau_1 \neq \tau_2$. $A_2(\tau_i) = (A_{21}(\tau_i), A_{22}(\tau_i))$ is a 1×2 row vector for $i = 1, 2$.

To explain the asset allocation puzzle, Lioui (2007) assumes that only the short rate is stochastic while the market prices are deterministic. For comparison, we follow Lioui (2007) to assume that the price of risk $\bar{\Lambda}(t)$ is a constant vector by setting $\bar{\lambda}_2 = 0$ and we can obtain, by equation (A3) of Appendix A in Sanjvinatsos and Wachter (2005), that

$$A_2(\tau) = \delta^\top K^{-1}(e^{-K\tau} - 1). \tag{28}$$

Denote the vectors of volatility and risk premia of the two bonds by

$$\sigma_P = \begin{pmatrix} A_2(\tau_1)\sigma_X \\ A_2(\tau_2)\sigma_X \end{pmatrix} = \begin{pmatrix} A_2(\tau_1) \\ A_2(\tau_2) \end{pmatrix} \sigma_X = A_2\sigma_X,$$

and $\mu_P = \sigma_P\bar{\Lambda}(t)$, respectively. In addition to the two bonds, we assume there exists an instantaneously riskless money market account with price at time t given by $B(t)$ and one stock index with price $S(t)$ where $B(t)$ and $S(t)$ satisfy

$$\frac{dB(t)}{B(t)} = r(t)dt, \tag{29}$$

$$\frac{dS(t)}{S(t)} = (\mu_S + r(t))dt + \sigma_S dZ(t) + JdN(t) - g^P\lambda^P dt, \tag{30}$$

where $\mu_S = \sigma_S \bar{\Lambda}(t) + g^P \lambda^P - g^Q \lambda^Q$; $\sigma_S = (\sigma_{S1}, \sigma_{S2})$; g^P and λ^P are the expected jump size and jump intensity under the physical measure P, respectively; g^Q and λ^Q are the expected jump size and jump intensity under the risk neutral measure Q, respectively. Specifically, μ_S is the total risk premium for the stock with the term $\sigma_S \bar{\Lambda}(t)$ compensating for the diffusion risk while the term $g^P \lambda^P - g^Q \lambda^Q$ compensating for the jump risk.

This specification implies the two bonds and cash are relatively safer than stock during a turbulent period when jump occurs. As is well understood, jumps in stock returns have significant impacts on the optimal portfolio choice. For instance, Liu, Longstaff, and Pan (2003) demonstrate that, in the presence of jumps in stock returns, an investor is less willing to take leveraged or short positions than in a standard diffusion model. Furthermore, even when the chance of a large jump is remote, an investor has strong incentives to significantly reduce her exposure to the stock market. The reason is that, if a jump occurs, invested wealth can change significantly from its current value, and such changes cannot be hedged through continuous rebalancing, resulting in potentially large losses for investors with leveraged or short positions. In stark contrast, changes in bond prices can be hedged through continuous rebalancing as they follow pure-diffusion processes. A natural question is: how does a risk averse investor choose her bond-stock mix when facing uncertain abrupt changes in stock returns? More concretely, does a more risk averse investor hold more bonds and/or cash than a less risk averse investor does? To answer these questions, we let π_{B1}^*, π_{B2}^* and π_S^* denote the fractions of the wealth invested in the two bonds and the stock, respectively. And hence, the remainder $\pi_C = 1 - \pi_{B1}^* - \pi_{B2}^* - \pi_S^*$ is invested in cash. The following proposition presents a closed-form solution to the optimal strategy.

Proposition 6 *The optimal portfolio weight $\pi^* = (\pi_{B1}^*, \pi_{B2}^*, \pi_S^*)$ is given by*

$$(\pi_{B1}^*, \pi_{B2}^*) = \left[\frac{\bar{\Lambda}(t)'}{\gamma} + \frac{(f_X)'}{f} \sigma_X \right] \sigma_p^{-1} - \tilde{\pi}_q \sigma_S \sigma_p^{-1} \tag{31}$$

$$\pi_S^* = \tilde{\pi}_q^*. \tag{32}$$

where the function $f(t, X_t)$ is given in Appendix E and $\widetilde{\pi}_q^*$ solves the following optimization problem:

$$\sup_{\widetilde{\pi}_q \in F} \widetilde{\pi}_q(-g^Q \lambda^Q) + \frac{\lambda^P}{1-\gamma} \int_A (1 + \widetilde{\pi}_q z)^{1-\gamma} \Phi(dz). \tag{33}$$

Proof. See Appendix D. ∎

Interestingly, Equation (32) shows that the demand for the stock index has a speculative component only to gain the risk premium from jumps as suggested by the static optimization problem for $\widetilde{\pi}_q^*$ while burden of hedging interest rate risk and and market price of risk is borne by the two bonds. It is worth mentioning that this result holds true regardless of whether or not $\tau_1 = T$, namely, the maturity of a bond must coincide with the investment horizon. The reason for the results in Proposition 6 is that the two bonds span the risk of both interest rate and market price of risk while only stock span the jump risk. By contrast, the bond portfolio weights have three components. The first is myopic demand for risk premia of two diffusion risks; the second is hedging demand to hedge against the risk stemming from the two diffusion risks; the third one is another myopic demand for jump risk premium. More specifically, as shown in Appendix E, the first two components are identical to the optimal weights in the market where the stock is not available for trading. And thus, the third component determines more or less bonds the investor holds when she can trades the stock. Although the two bonds are independent from the jumps, the investor can gain the jump risk premium by investing more in the two bonds. The reason is that the two bonds and the stock are correlated via diffusion as this can be seen from the term $\sigma_S \sigma_p^{-1}$.

To make the intuition behind the results as clear as possible, we concentrate on a simple case by further assuming that $\delta_0 = 0$ and $\delta^\top = (1, 1)$; the jump size $J = g^P$ is a negative constant under the physical measure P; the state variables X_1 and X_2 are independent with $\sigma_{X_{12}} = \sigma_{X_{21}} = 0$, namely, X_1 and X_2 follow the equations below.

$$dX_1(t) = K_1(\theta_1 - X_1(t))dt + \sigma_{X_{11}}dZ_1(t),$$

$$dX_2(t) = K_2(\theta_2 - X_2(t))dt + \sigma_{X_{22}}dZ_2(t),$$

where K_1 and K_2 are positive constants. In this case, by (28), we have

$$A_{2i}(\tau) = \frac{e^{-K_i \tau} - 1}{K_i}, i = 1, 2.$$

We further assume that X_1 is a permanent state variable with a low K_1 while X_2 is a transitory state variable with a high K_2. Like Table II in Sanjvinatsos and Wachter (2005), we let $\sigma_{X_{11}} > 0, \sigma_{X_{22}} > 0, \sigma_{S1} < 0, \sigma_{S2} > 0$ and $\sigma_{S1}\sigma_{X_{11}} + \sigma_{S2}\sigma_{X_{22}} < 0$ such that the stock return is negatively correlated with both state variable $X_1(t)$ and interest rate $r(t)$. From (27), it is easy to check that the bond return and the interest rate has negative correlation as $A_{21}(\tau) < 0$ and $A_{22}(\tau) < 0$. Furthermore, in order to investigate whether or not the explanation of Lioui (2007) for the bond/stock ratio puzzle is still valid in our jump-diffusion model, we assume that the maturity τ_1 of the first bond is equal to the investment horizon T. Then, the optimal portfolio weights in Proposition 7 are given explicitly in the following result.

Proposition 7 *The optimal portfolio weight $\pi^* = (\pi_{B1}^*, \pi_{B2}^*, \pi_S^*)$ is given by*

$$\pi_{B1}^* = \frac{1}{\gamma |A_2|}\left(\frac{\bar{\Lambda}_1(t)}{\sigma_{X_{11}}}A_{22}(\tau_2) - \frac{\bar{\Lambda}_2(t)}{\sigma_{X_{22}}}A_{21}(\tau_2)\right) + \left(1 - \frac{1}{\gamma}\right) - \frac{\tilde{\pi}_q^*}{|A_2|}\left(\frac{\sigma_{S1}}{\sigma_{X_{11}}}A_{22}(\tau_2) - \frac{\sigma_{S2}}{\sigma_{X_{22}}}A_{21}(\tau_2)\right)$$

(34)

$$\pi_{B2}^* = \frac{1}{\gamma |A_2|}\left(-\frac{\bar{\Lambda}_1(t)}{\sigma_{X_{11}}}A_{22}(\tau_1) + \frac{\bar{\Lambda}_2(t)}{\sigma_{X_{22}}}A_{21}(\tau_1)\right) - \frac{\tilde{\pi}_q^*}{|A_2|}\left(-\frac{\sigma_{S1}}{\sigma_{X_{11}}}A_{22}(\tau_1) + \frac{\sigma_{S2}}{\sigma_{X_{22}}}A_{21}(\tau_1)\right)$$

(35)

$$\pi_S^* = \tilde{\pi}_q^* = \frac{1}{g^P}\left[\left(\frac{g^Q \lambda^Q}{g^P \lambda^P}\right)^{-\frac{1}{\gamma}} - 1\right],$$

(36)

where $|A_2| = A_{21}(\tau_1)A_{22}(\tau_1) - A_{21}(\tau_2)A_{22}(\tau_1)$.

Proof. See Appendix D. ∎

The above results suggest that Bond 1 perfectly hedges the interest rate risk, which is

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the same as a pure-diffusion model in Lioui (2007). We can show that $|A_2| < 0$ provided that $K_1 < K_2$ and $\tau_1 < \tau_2$. Using the facts that $A_{21}(\tau) < 0$, $A_{22}(\tau) < 0$ and $\sigma_{S1} < 0$, we can verify that the coefficient of $\tilde{\pi}_q^*$ in Equation (34) is positive while the one in Equation (35) is negative. In other words, to gain jump risk premium, the investor holds more short term bond (Bond 1) and less long-term bond (Bond 2) to offset the position in Bond 1. In the meantime, the total demand for the two bonds due to the jump is positive as we can write this demand as

$$-\frac{\tilde{\pi}_q^*}{|A_2|}\left[\frac{\sigma_{S1}}{\sigma_{X_{11}}}(A_{22}(\tau_2) - A_{22}(\tau_1)) + \frac{\sigma_{S2}}{\sigma_{X_{22}}}(A_{21}(\tau_1) - A_{21}(\tau_2))\right] \tag{37}$$

and the coefficient of $\tilde{\pi}_q^*$ is positive.

We now turn to the impact of risk aversion coefficient γ on the bond/stock mix. From Proposition 7, the bond/stock ratio is separated into three terms which correspond to three parts in the portfolio on the bonds: mean-variance allocation, hedging demand for interest risk, and myopic demand for jump risk. The second term is actually exploited to explain the asset allocation puzzle in literature (see e.g. Bajeux-Besnainou, et al. (2001) and Lioui (2007)). It is interesting to investigate whether the ratio increases with the relative risk aversion coefficient γ in our model here. To this purpose, we rewrite the total demand for the two bonds in Proposition 7 as:

$$\pi_B^* = \frac{a}{\gamma} + 1 - \frac{1}{\gamma} - b\tilde{\pi}_q^*,$$

with

$$a = \frac{1}{|A_2|}\left[\frac{\bar{\Lambda}_1(t)}{\sigma_{X_{11}}}(A_{22}(\tau_2) - A_{22}(\tau_1)) + \frac{\bar{\Lambda}_2(t)}{\sigma_{X_{22}}}(A_{21}(\tau_1) - A_{21}(\tau_2))\right],$$

$$b = \frac{1}{|A_2|}\left[\frac{\sigma_{S1}}{\sigma_{X_{11}}}(A_{22}(\tau_2) - A_{22}(\tau_1)) + \frac{\sigma_{S2}}{\sigma_{X_{22}}}(A_{21}(\tau_1) - A_{21}(\tau_2))\right].$$

And hence, the bond/stock ratio is obtained as:

$$f(\gamma) = \frac{\pi_B^*}{\tilde{\pi}_q^*} = \left(\frac{a-1}{\gamma} + 1\right)\frac{1}{\tilde{\pi}_q^*} - b,$$

implying

$$f'(\gamma) = \frac{df(\gamma)}{d\gamma} = \frac{1}{\gamma^2 \tilde{\pi}_q^*}\left[1 - a - \frac{1}{g^P \tilde{\pi}_q^*}\left(\frac{a-1}{\gamma} + 1\right)\left(\frac{g^Q \lambda^Q}{g^P \lambda^P}\right)^{-\frac{1}{\gamma}} ln\left(\frac{g^Q \lambda^Q}{g^P \lambda^P}\right)\right].$$

It can be seen the function $f'(\gamma)$ can be either positive or negative depending on the model parameters. For instance, we show that it can be negative under certain conditions. For this, we rewrite $f'(\gamma)$ as

$$f'(\gamma) = \frac{1}{\gamma^2 \tilde{\pi}_q^*}\left[1 - a - \left(\frac{a-1}{\gamma} + 1\right)\frac{ln\left(\frac{g^Q \lambda^Q}{g^P \lambda^P}\right)}{1 - \left(\frac{g^Q \lambda^Q}{g^P \lambda^P}\right)^{\frac{1}{\gamma}}}\right].$$

Considering the case: $1 \le \gamma \le 3$ and $a > 1$, we can show that $f'(\gamma) < 0$ when $\frac{g^Q \lambda^Q}{g^P \lambda^P} > \frac{a+2}{a-1}$, that is, the jump risk premium is high enough. Therefore, in this case, the ratio $\frac{\pi_B^*}{\tilde{\pi}_q^*}$ is a decreasing function of γ in the range of $[1,3]$. The reason for this is that unlike a pure-diffusion model, the demand $\tilde{\pi}_q^*$ for the stock is not proportional to $1/\gamma$. In fact, $\tilde{\pi}_q^*$ decreases slower than $1/\gamma$ when γ increases. This is in stark contrast with pure-diffusion model. Specifically, our jump-diffusion model reduces to a pure-diffusion model by replacing the jump in stock return with a diffusion $Z_3(t)$. Then, the results in Proposition 8 except π_S^* remain unchanged. In the meantime, $\pi_S^* = C/\gamma$ for a positive constant C. As a result, $f(\gamma) = (a - 1 + \gamma)\frac{1}{C} - b$, which is an increasing function of γ. And thus, as in Lioui(2007), this leads to the resolution of the asset allocation puzzle in the pure-diffusion model. In short, the rationality of the bond/stock ratio puzzle cannot be explained by the intertemporal hedging demand in the presence of jumps in the stock return, and thus our jump-diffusion model provides another channel to strengthen the issue raised by Lioui (2007), that is, the asset allocation is still a puzzle.

Finally, we investigate the effects of the jump parameters on the cash-bond-stock mix.

For simplicity, let us consider the jump intensity λ^P. Note that from (36),

$$\frac{\partial \tilde{\pi}_q^*}{\partial \lambda^P} = \frac{1}{\gamma g^P}(\lambda^P)^{\frac{1}{\gamma}-1}\Big(\frac{g^Q \lambda^Q}{g^P}\Big)^{-\frac{1}{\gamma}} < 0.$$

Hence, the investor holds less stocks when facing more frequent jumps. In other words, the investor reduces her position in the stock during a turbulent time of stock market. In the meantime, she also reduces her bond holding based on the above discussion based on (37). As a result, the investor holds more cash, reflecting the phenomenon of flight-to-safety.

5   Conclusion

In the present paper, we obtain closed-form solutions the optimal dynamic portfolio selection problem in multi-asset affine jump-diffusion models where both stock returns and state variables may exhibit time-varying jumps. More specifically, our closed-form formulas for the indirect value function and the optimal portfolio weights are in terms of the solutions to a set of ODEs. Our results extend the pure-diffusion models in Liu (2007) by incorporating jumps in both stock returns and state variables. Our results also extend those in Jin and Zhang (2012) by including jumps in state variables in affine jump-diffusion settings and solving the optimal portfolio choice problem without simulation.

We focus on two application. In the first application, we propose a tractable model to explicitly solve the optimal investment problem in variance swaps. The second application shows that unlike in pure-diffusion models, there is no clear-cut answer to the bond/stock ratio puzzle in a jump-diffusion model despite the hedging assumption.

Appendix A-D

A   Proof of Proposition 1 and Proposition 2

The proof is inspired by the proof of Proposition 1 in Jin and Zhang (2012). In particular, in exactly the same manner as the one in the proof of Proposition 1 in Jin and Zhang (2012), we can show that $\widetilde{\pi}_{qk}^*$ is the optimal solution to the problem

$$\max_{\widetilde{\pi}_{qk} \geq 0} W^{1-\gamma}(f(X,t))^\gamma \left(\widetilde{\pi}_{qk}(\theta_k^q - \lambda_k a_k) + \frac{\lambda_k}{1-\gamma} E\left[(1 + \widetilde{\pi}_{qk} Y_k^s)^{1-\gamma} e^{B(t)\sigma_{Jk}^x Y_k^x} \right] \right).$$

By noticing that $\theta_k^q = \theta_k^0 \lambda_k$ and $\theta_k^q - \lambda_k a_k = (\theta_k^0 - a_k)\lambda_k$, the above optimization problem is is identical to the following problem:

$$\max_{\widetilde{\pi}_{qk} \geq 0} \left(\widetilde{\pi}_{qk}(\theta_k^0 - a_k) + \frac{1}{1-\gamma} E\left[(1 + \widetilde{\pi}_{qk} Y_k^s)^{1-\gamma} e^{B(t)\sigma_{Jk}^x Y_k^x} \right] \right),$$

and thus the optimal jump exposure $\widetilde{\pi}_{qk}^*$ is deterministic since there are no state variables in the above optimization problem. For the optimal diffusion exposure $\widetilde{\pi}_b^*$, from the proof of Proposition 1 in Jin and Zhang (2012), it is given by

$$\widetilde{\pi}_b^{*\top} = \frac{\theta^b}{\gamma} + \rho_t^\top \sigma^{x\top} \frac{f_X}{f} = \frac{\theta^b}{\gamma} + \rho_t^\top \sigma^{x\top} B^\top(t).$$

Furthermore, according to the proof of Proposition 2 in Jin and Zhang (2012), given the optimal jump exposures $\widetilde{\pi}_{qk}^*, k = 1, ..., n - d$, the diffusion exposure vector $\widetilde{\pi}_b^*$ solves the following problem

$$\begin{aligned}
0 = \max_{\widetilde{\pi}_b} \Bigg\{ &J_t + \frac{1}{2} W^2 \widetilde{\pi}_b \widetilde{\pi}_b^\top J_{WW} + W(\widetilde{\pi}_b \theta^b + r) J_W \\
&+ b^x J_X + W \widetilde{\pi}_b \rho_t^\top \sigma^{x\top} J_{WX} + \frac{1}{2} Tr(\sigma^x \sigma^{x\top} J_{XX^\top}) + \sum_{k=1}^{n-d} \lambda_k D_k J \Bigg\}.
\end{aligned}$$

Then, substituting the indirect value function $J(t, W, X_t) = \frac{W^{1-\gamma}}{1-\gamma} \left[e^{A(t)+B(t)X_t} \right]^\gamma$ and the optimal diffusion exposure vector $\widetilde{\pi}_b^*$ into the above equation and matching coefficients gives

the ODEs for the functions $A(t)$ and $B(t)$ specified in Proposition 1, completing the proof.

∎

B   Proof of Proposition 3 and Proposition 4

We apply Proposition 1 for the proof of Proposition 3. The state variables in this model are v_t, m_t and λ_t. Note that in this case

$$b - r\mathbf{1}_3 = \begin{pmatrix} \phi_v(\tau_1)\sigma_v\gamma_2 v_t + \phi_m(\tau_1)\sigma_m\gamma_3 m_t - (\phi_v(\tau_1) + \beta_0\phi_\lambda(\tau_1))\mu_v^Q\lambda_t \\ \phi_v(\tau_2)\sigma_v\gamma_2 v_t + \phi_m(\tau_2)\sigma_m\gamma_3 m_t - (\phi_v(\tau_2) + \beta_0\phi_\lambda(\tau_2))\mu_v^Q\lambda_t \\ \phi_v(\tau_3)\sigma_v\gamma_2 v_t + \phi_m(\tau_3)\sigma_m\gamma_3 m_t - (\phi_v(\tau_3) + \beta_0\phi_\lambda(\tau_3))\mu_v^Q\lambda_t \end{pmatrix} = \Sigma \begin{pmatrix} \gamma_2\sqrt{v_t} \\ \gamma_3\sqrt{m_t} \\ -\mu_v^Q\lambda_t \end{pmatrix}.$$

Hence

$$\theta = \begin{pmatrix} \theta_1^b \\ \theta_2^b \\ \theta_1^q \end{pmatrix} = \Sigma^{-1}(b - r\mathbf{1}_3 + \Sigma_q E^P[J_t^{v,P}]\lambda_t) = \begin{pmatrix} \gamma_2\sqrt{v_t} \\ \gamma_3\sqrt{m_t} \\ -(E^Q[J_t^{v,P}] - E^P[J_t^{v,P}])\lambda_t \end{pmatrix}.$$

By noticing that the state variable λ_t is a pure-jump process, applying Proposition 1 gives the following indirect value function:

$$J(t, W_t, X_t) = \frac{W_t^{1-\gamma}}{1-\gamma}[f(t, X_t)]^\gamma = \frac{W^{1-\gamma}}{1-\gamma}\left[e^{A(t)+B_1(t)v_t+B_2(t)m_t+B_3(t)\lambda_t}\right]^\gamma \tag{38}$$

where the functions $A(t)$, $B(t) = (B_1(t), B_2(t))^\top$ and $B_3(t)$ satisfy the following equations:

$$\frac{dA}{dt} + \left(k + \frac{1-\gamma}{\gamma}g_0\right)^\top B^\top + \frac{1}{2}B[h_0 + (1-\gamma)l_0]B^\top$$
$$+ \frac{1-\gamma}{2\gamma^2}H_0 + \frac{1-\gamma}{\gamma}\delta_0 = 0,$$
$$\frac{dB}{dt} + \left(-K + \frac{1-\gamma}{\gamma}g_1\right)^\top B^\top + \frac{1}{2}B[h_1 + (1-\gamma)l_1]B^\top$$
$$+ \frac{1-\gamma}{2\gamma^2}H_1 + \frac{1-\gamma}{\gamma}\delta_1 = 0$$
$$\frac{dB_3}{dt} - \alpha B_3 + (\gamma-1)\widetilde{\pi}_{q1}^* E^Q[J^{v,Q}] + E^P\left[\left(\widetilde{\pi}_{q1}^* J^{v,P} + 1\right)^{1-\gamma}e^{\gamma(B_1+B_3\beta_0)J^{v,P}} - 1\right] = 0,$$

Furthermore, by noticing that

$$
\sigma^x(X_t) = \begin{pmatrix} \sigma_v \sqrt{v_t} & 0 \\ 0 & \sigma_m \sqrt{m_t} \end{pmatrix},
$$

we can get the following parameters: $k = (0, \kappa_m^P \theta_m^P)^\top$, $h_{111} = (\sigma_v^2, 0), h_{112} = h_{121} = (0, 0), h_{122} = (0, \sigma_m^2), \delta_0 = r, \delta_1 = 0, H_0 = 0, H_1 = (\gamma_2^2, \gamma_3^2), g_0 = 0, l_0 = 0, l_1 = 0.$

$$
K = \begin{pmatrix} \kappa_v^P & -\kappa_v^Q \\ 0 & \kappa_m^P \end{pmatrix}, g_1 = \begin{pmatrix} \sigma_v \gamma_2 & 0 \\ 0 & \sigma_m \gamma_3 \end{pmatrix},
$$

Proposition 3 and Proposition 4 will follow from results in Proposition 1 and Proposition 2. ∎

C Proof of Proposition 5

We assume that the model "SV2F-PJ-VJ" in Ait-Sahalia et al. (2015) is the true model and then evaluate the utility costs of suboptimal strategies based on the models "SV2F" and "SV2F-PJ", respectively. The indirect value function J of the true model "SV2F-PJ-VJ" is given in Proposition 3. We now first derive the indirect utility corresponding to the model "SV2F". Suppose that the two variance swaps have maturities of τ_1 and τ_2. Let n_{1t} and n_{2t} denote the optimal portfolio strategy in the model "SV2F" which are given by (43) and (44) in Proposition 4 of Egloff et al. (2010). Consider the corresponding strategy $w = (n_{1t}, n_{2t})^\top$ in the the model "SV2F-PJ-VJ" and let $J^{(1)}$ denote its indirect utility. Then $J^{(1)}$ satisfies the equation below

$$
\begin{aligned}
0 =\ & J_t^{(1)} + \frac{1}{2} W^2 w^\top \Sigma_b^{(1)} (\Sigma_b^{(1)})^\top w J_{WW}^{(1)} + W(w^\top (b - r\mathbf{1}_2) + r) J_W^{(1)} \\
& + (b^x)^\top J_X^{(1)} + \alpha(\lambda_\infty - \lambda_t) J_\lambda^{(1)} + W w^\top \Sigma_b^{(1)} \sigma^{x\top} J_{WX}^{(1)} + \frac{1}{2} Tr(\sigma^x \sigma^{x\top} J_{XX^\top}^{(1)}) \\
& + \lambda_t E^P \left[J^{(1)} \left(W \left(\tilde{\pi}_{q1} J^{v,P} + 1 \right), v_t + J^{v,P}, m_t, \lambda_t + \beta_0 J^{v,P} \right) - J^{(1)}(W, v_t, m_t, \lambda_t) \right],
\end{aligned} \quad (39)
$$

where $X_t = (v_t, m_t)^\top$,

$$\widetilde{\pi}_{q1} = n_{1t}(\phi_v(\tau_1) + \beta_0\phi_\lambda(\tau_1)) + n_{2t}(\phi_v(\tau_2) + \beta_0\phi_\lambda(\tau_2)),$$

$$\Sigma_b^{(1)} = \begin{pmatrix} \phi_v(\tau_1)\sigma_v\sqrt{v_t} & \phi_m(\tau_1)\sigma_m\sqrt{m_t} \\ \phi_v(\tau_2)\sigma_v\sqrt{v_t} & \phi_m(\tau_2)\sigma_m\sqrt{m_t} \end{pmatrix}$$

$$b - r\mathbf{1}_2 = \begin{pmatrix} \phi_v(\tau_1)\sigma_v\gamma_2 v_t + \phi_m(\tau_1)\sigma_m\gamma_3 m_t - (\phi_v(\tau_1) + \beta_0\phi_\lambda(\tau_1))\mu_v^Q\lambda_t \\ \phi_v(\tau_2)\sigma_v\gamma_2 v_t + \phi_m(\tau_2)\sigma_m\gamma_3 m_t - (\phi_v(\tau_2) + \beta_0\phi_\lambda(\tau_2))\mu_v^Q\lambda_t \end{pmatrix}.$$

We guess the following indirect utility function:

$$J^{(1)}(t, W_t, X_t) = \frac{W_t^{1-\gamma}}{1-\gamma}[f^{(1)}(t, X_t)]^\gamma = \frac{W^{1-\gamma}}{1-\gamma}\left[e^{A^{(1)}(t)+B_1^{(1)}(t)v_t+B_2^{(1)}(t)m_t+B_3^{(1)}(t)\lambda_t}\right]^\gamma \quad (40)$$

By using the same method as in the proof of Proposition 1, substituting the above function $J^{(1)}$ into the equation (39) gives the ODEs for the functions $A^{(1)}(t)$, $B^{(1)}(t) = (B_1^{(1)}(t), B_2^{(1)}(t))^\top$ and $B_3^{(1)}(t)$ in Proposition. Furthermore, the utility cost, CE, of following the suboptimal strategy $w = (n_{1t}, n_{2t})^\top$ is obtained by applying the formulas of $J(t, W_t(1 - CE), X_t)$ and $J^{(1)}(t, W_t, X_t)$. $\blacksquare$

D Proof of Propositions 6 and 7

We first prove Proposition (6). For portfolio strategy $\pi_t = (\pi_{B1}, \pi_{B2}, \pi_S)^\top$, the wealth process is given by

$$\frac{dW_t}{W_t} = \pi'(\mu - r)dt + r(t)dt + \pi'\sigma dZ(t) + \pi_S JdN_t,$$

where

$$\mu = \begin{pmatrix} \mu_P \\ \mu_S - g^P \lambda^P \end{pmatrix}, \sigma = \begin{pmatrix} \sigma_P \\ \sigma_S \end{pmatrix},$$

Define $\Sigma = [\sigma, \sigma_J]$ with $\sigma_J = (0,0,1)^\top$. Note that μ can be rewritten as $\mu - r\mathbf{1}_3 = \Sigma(\bar{\Lambda}_1(t), \bar{\Lambda}_2(t), -g^Q \lambda^Q)^\top$ and

$$\Sigma^{-1} = \begin{pmatrix} \sigma_p^{-1} & 0_{2 \times 1} \\ -\sigma_S \sigma_p^{-1} & 1 \end{pmatrix} \tag{41}$$

Hence, from (4),

$$(\theta_1^b, \theta_2^b, \theta^q)^\top = \Sigma^{-1}(\mu - r\mathbf{1}_3 + \sigma_J \lambda^P g^P) = (\bar{\Lambda}_1(t), \bar{\Lambda}_2(t), -\lambda^Q g^Q + \lambda^P g^P)^\top. \tag{42}$$

Then we can obtain the optimal portfolio choice problem and its solution by Proposition 4:

$$(\pi_{B1}, \pi_{B2}, \pi_S) = (\tilde{\pi}_{b1}^*, \tilde{\pi}_{b2}^*, \tilde{\pi}_q^*)\Sigma^{-1} \tag{43}$$

where $(\tilde{\pi}_{b1}^*, \tilde{\pi}_{b2}^*)^\top = \frac{\bar{\Lambda}(t)}{\gamma} + (\sigma_X)' \frac{f_X}{f}$, where $f(X)$ is given by Proposition 1. In particular, $\tilde{\pi}_q^*$ solves the following optimization problem.

$$\sup_{\tilde{\pi}_q \geq 0} -g^Q \lambda^Q \tilde{\pi}_q + \frac{\lambda^P}{1 - \gamma} \int_A (1 + \tilde{\pi}_q z)^{1-\gamma} \Phi(dz). \tag{44}$$

By Proposition 1, $f(t, X_t)$ can be written as: $f(x,t) = e^{A(t) + B(t)x}$, where $A(t) \in R$, $B(t) \in R^{1 \times 2}$, $A(T) = 0$ and $B(T) = 0$. Then it follows that

$$\frac{dA}{dt} + \left(k + \frac{1 - \gamma}{\gamma} g_0\right)^\top B^\top(t) + \frac{1}{2} B(t) h_0 B^\top(t)$$

$$+ \frac{1 - \gamma}{2\gamma^2} H_0 + \frac{1 - \gamma}{\gamma} \delta_0 + \lambda_0^P D = 0,$$

$$\frac{dB(t)}{dt} - K^\top B^\top(t) + \frac{1 - \gamma}{\gamma} \delta^\top = 0,$$

where $k = K\theta$, $g_0 = \sigma_X \bar{\Lambda}(t)^\top$, $h_0 = \sigma_X \sigma_X^\top$, $H_0 = \bar{\Lambda}(t)\bar{\Lambda}(t)^\top$, and

$$D = \frac{1-\gamma}{\gamma}\left(-g^Q \lambda^Q \tilde{\pi}_q^* + \frac{\lambda^P}{1-\gamma}\int_A (1 + \tilde{\pi}_q^* z)^{1-\gamma}\Phi(dz)\right). \tag{45}$$

Then, by solving the ODE for $B(t)$, we obtain $B(t) = (1 - \frac{1}{\gamma})\delta' K^{-1}(e^{-K\tau} - 1) = (1 - \frac{1}{\gamma})A_2(\tau)$. and hence, $(\tilde{\pi}_{b1}^*, \tilde{\pi}_{b2}^*)^\top = \frac{\bar{\Lambda}(t)}{\gamma} + (\sigma_X)'\frac{f_X}{f} = \frac{\bar{\Lambda}(t)}{\gamma} + (\sigma_X)'B^\top(t)$. Therefore,

$$(\pi_{B1}, \pi_{B2}) = \left[\frac{\bar{\Lambda}'(t)}{\gamma} + B(t)\sigma_X\right]\sigma_p^{-1} - \tilde{\pi}_q \sigma_S \sigma_P^{-1} \tag{46}$$

$$\pi_S = \tilde{\pi}_q. \tag{47}$$

Given the function f, the first term on the right hand side of (46) is the optimal portfolio weights in the two bonds in the market where the stock is not available for trading.

Hence, using the fact that $\sigma_P^{-1} = \sigma_X^{-1}A_2^{-1}$, we have $B(t)\sigma_X\sigma_P^{-1} = (1 - \frac{1}{\gamma})A(\tau)\sigma_X\sigma_X^{-1}A_2^{-1} = (1 - \frac{1}{\gamma})A_2(\tau)A_2^{-1}$, completing the proof of Proposition 6.

We now turn to the proof of Proposition 7. The first-order condition for the optimization problem (33) is given by

$$-g^Q \lambda^Q + g^P \lambda^P (1 + \tilde{\pi}_q g^P)^{-\gamma} = 0,$$

implying

$$\tilde{\pi}_q = \frac{1}{g^P}\left[\left(\frac{g^Q \lambda^Q}{g^P \lambda^P}\right)^{-\frac{1}{\gamma}} - 1\right].$$

Given $\sigma_{X_{12}} = \sigma_{X_{21}} = 0$, we have

$$\sigma_X^{-1} = \begin{pmatrix} \frac{1}{\sigma_{X_{11}}} & 0 \\ 0 & \frac{1}{\sigma_{X_{22}}} \end{pmatrix} \tag{48}$$

An explicit calculation for A_2^{-1} gives

$$A_2^{-1} = \frac{1}{|A_2|} \begin{pmatrix} A_{22}(\tau_2) & -A_{22}(\tau_1) \\ -A_{21}(\tau_2) & A_{21}(\tau_1) \end{pmatrix} \tag{49}$$

In particular, if the terminal horizon is the same as one of maturity dates, say, $\tau_1 = T$, then $A_2(\tau)A_2^{-1} = (1,0)^\top$. Substituting results above into Proposition 6 yields the desired result in Proposition 7. This completes the proof. ∎
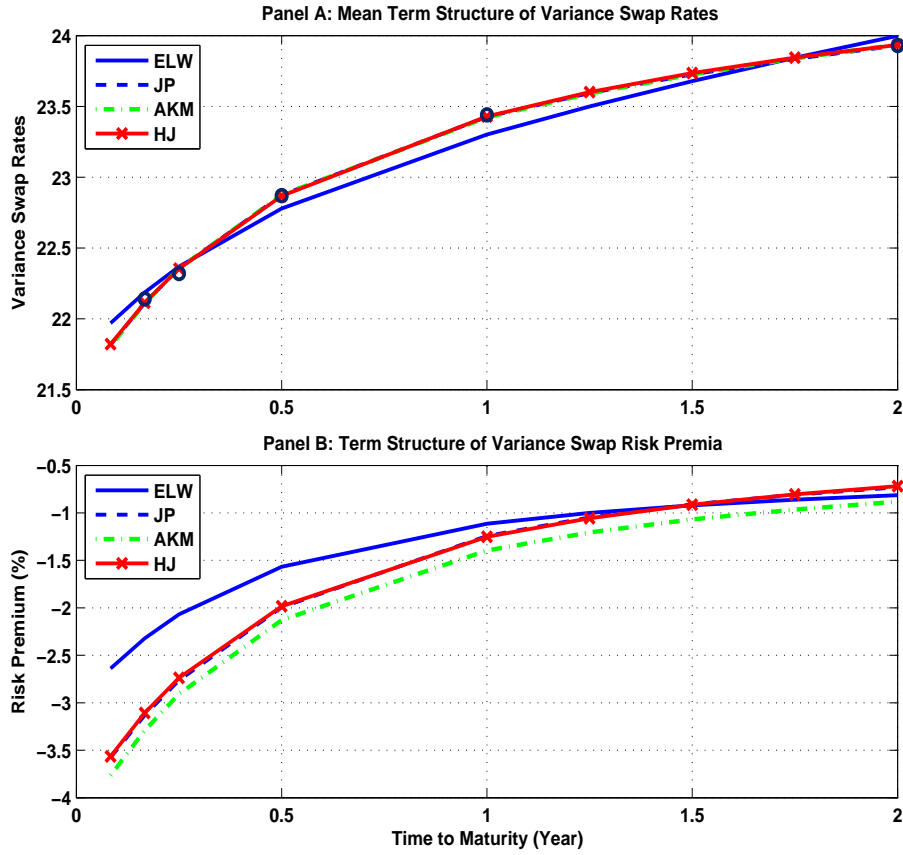
Figure 3: **Mean Term Structure of Variance Swap Rates and Risk Premia.** In the top panel, the mean term structure of variance swap rates produced by the ELW,JP, AKM and HJ model is denoted by the dashed line, the dotted line, the solid line and the solid line with "x", respectively, while the empirical means of the variance swap rates with the five time-to-maturities reported in Table 1 are represented by "∘". In the bottom panel, the term structure of risk premia in variance swap contracts that are compensated for risk factors in these four models is denoted by the lines with the same format accordingly. Also, the resulted RMSEs are reported as follows:

| | Time to Maturity | | | | | |
	2	3	6	12	24	RMSE
Empirical Mean	22.14	22.32	22.87	23.44	23.93	-
ELW	22.19	22.37	22.78	23.30	24.00	0.0858
JP	22.11	22.36	22.87	23.43	23.93	0.0210
AKM	22.11	22.36	22.87	23.42	23.94	0.0230
HJ	22.11	22.35	22.87	23.43	23.93	0.0199

Figure 4: **Optimal Investment in Variance Swap Contracts in ELW Model.** The optimal investments in variance swap contracts (as the fractions of total wealth in notional) in the ELW model (with $\gamma = 5$) is plotted as a function of the market price of the variance risk (γ_2) and the market price of the central tendency factor (γ_3). Panel A shows the optimal investments in 2-month and 2-year variance swap contracts, while Panel B shows the investment in 6-month and 1-year contracts. The surface on the top in each panel denotes the holdings of the long-term contract with maturity τ_2, while the surface on the below shows the holdings of the short-term contract with maturity τ_1. The investment horizon is set as two months (e.g., $T = 2$ months).
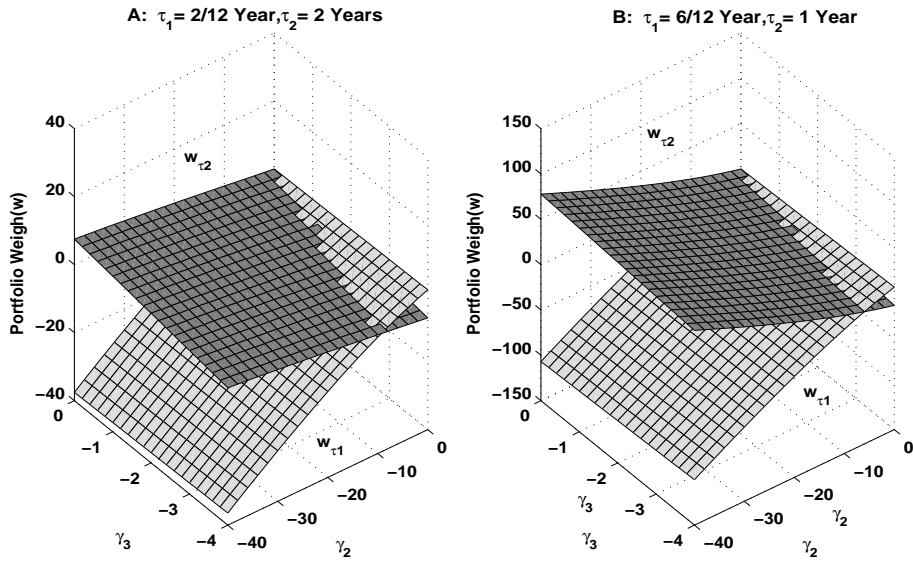
Figure 5: **Optimal Investment in Variance Swap Contracts in HJ Model.** The optimal investments in variance swap contracts (as the fractions of total wealth in notional) in the HJ model (with $\gamma = 5$ and $T = 2$ months) are plotted as a function of the market price of the variance risk (γ_2) and the market price of the central tendency factor (γ_3). When the maturities of the first two contracts, including the 2-month and 2-year variance swap contract in Panel A and the 6-month and 1-year contract in Panel B, C and D, are specified, the four panels show the optimal investments in three contracts by positioning the third one with different maturity. In each panel, the surface on the top denotes the holdings of the medium-term contract, and the surface on the below shows the holdings of the long-term contract, while the surface in the middle presents the holdings of the short-term contract.
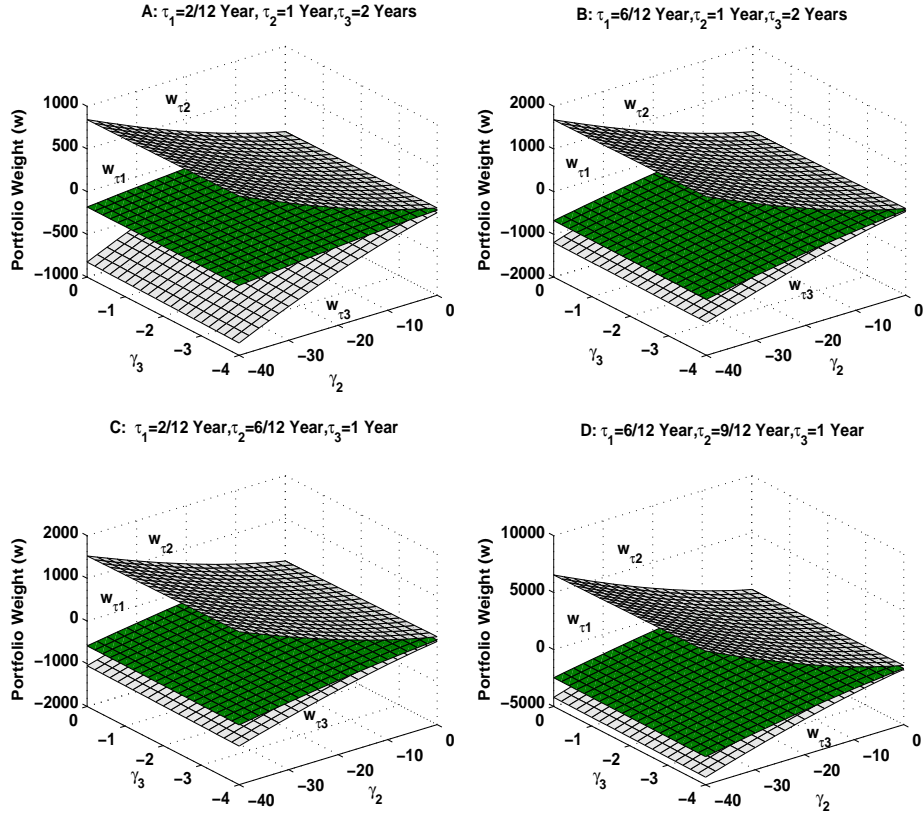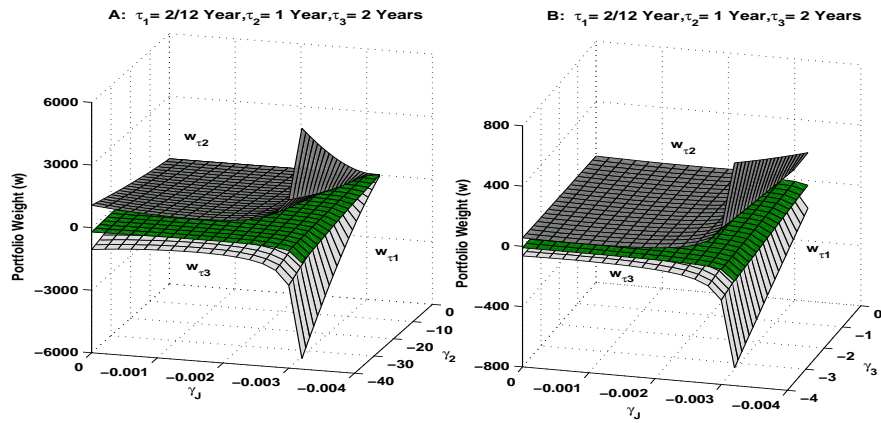
Figure 6: **Sensitivity of Optimal Investments in Variance Swap Contracts to Jump Risk in HJ Model.** The optimal investment in variance swap contracts (as the fractions of total wealth in notional) in the HJ model (with $\gamma = 5$ and $T = 2$ months) is plotted as a function of the market price of the variance risk (γ_2), the market price of the central tendency factor (γ_3) and the market price of the jump risk (γ_J). Three contracts are traded, including the 2-month, 1-year and 2-year variance swap contract (denoted by τ_1, τ_2 and τ_3 respectively). In each panel, the surface on the top denotes the holdings of the maturity-τ_2 contract (the medium-term contract), and the surface on the below shows the holdings of the long-term contract with maturity τ_3, while the surface in the middle presents the holdings of the short-term contract with maturity τ_1.

| RRA | Investment Horizon (T) (Year) | | | | | Myopic Portfolio |
	0.5	1	5	10	20	(M)

Panel I: Hedging Demands for Volatility (H_v/M)

RRA	0.5	1	5	10	20	(M)
	0.040	0.043	0.043	0.043	0.043	-15.862
2	0.039	0.042	0.042	0.042	0.042	75.431
	0.038	0.041	0.041	0.041	0.041	-74.193
	0.066	0.073	0.073	0.073	0.073	-6.345
5	0.064	0.071	0.071	0.071	0.071	30.172
	0.063	0.069	0.070	0.070	0.070	-29.677
	0.083	0.091	0.092	0.092	0.092	-0.793
40	0.081	0.089	0.090	0.090	0.090	3.772
	0.079	0.087	0.088	0.088	0.088	-3.710

Panel II: Hedging Demands for Central Tendency (H_m/M)

RRA	0.5	1	5	10	20	(M)
	0.005	0.010	0.030	0.032	0.033	-15.862
2	0.010	0.023	0.065	0.071	0.072	75.431
	0.014	0.032	0.091	0.099	0.100	-74.193
	0.008	0.018	0.051	0.057	0.057	-6.345
5	0.017	0.039	0.112	0.124	0.125	30.172
	0.023	0.054	0.157	0.173	0.175	-29.677
	0.009	0.022	0.066	0.074	0.075	-0.793
40	0.021	0.048	0.145	0.162	0.164	3.772
	0.029	0.067	0.202	0.226	0.228	-3.710

Table 3: **Hedging Ratios for Volatility (v) and Central Tendency (m).** The hedging ratios for volatility v and central tendency m are calculated using the hedging demands in Equation (20) with various risk aversion of the trader: the less risk aversion ($\gamma = 2$), the moderate risk aversion ($\gamma = 5$) and the extreme risk aversion ($\gamma = 40$). The variance swap contracts with typical maturities are used, including a set of the 2-month, 1-year and 2-year variance swap contracts. Each entry of the array in both panels consists of three components: the first of which is the hedging ratio for the 2-month contract (τ_1), the second one for the 1-year contract (τ_2) and the third one for the 2-year contract (τ_3), respectively.
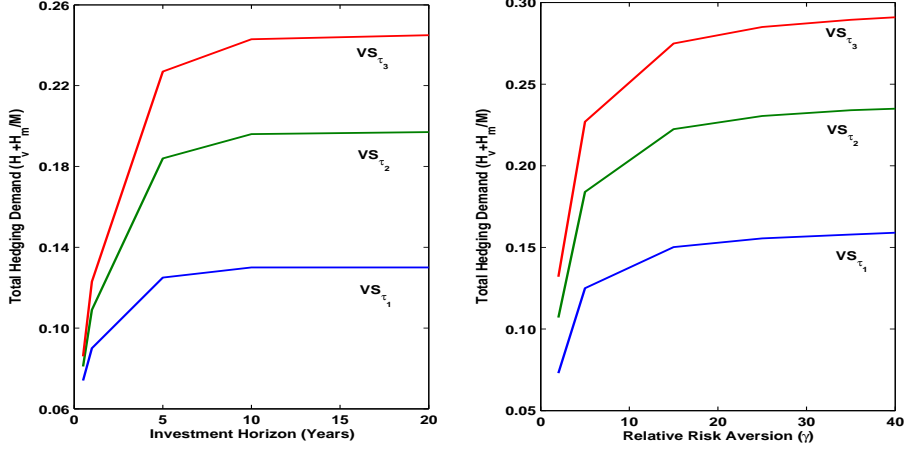
Figure 7: **Effects of Investment Horizon (T) and Risk Aversion (γ) on Total Hedging Demand in HJ Model.** The total hedging demand for volatility v and central tendency m, reported as a percentage of the myopic portfolio (e.g., $(H_v + H_m)/M$), are plotted as a function of the investment horizon (T) in the left panel and of the risk aversion (γ) in the right panel. In the left panel, the degree of risk aversion is set as $\gamma = 5$, while a fixed investment horizon is set as $T = 5$ years in the right panel. The variance swap contracts with typical maturities are used, including a set of the 2-month, 1-year and 2-year variance swap contracts, represented by VS_{τ_1}, VS_{τ_2} and VS_{τ_3}, respectively.

RRA	Maturity Pair (Year)	Investment Horizon (T) (Year)					
		0.5	1	5	10	20	30
$\gamma = 5$	$(\tau_1 = 2/12, \tau_2 = 2)$	-1.32	-1.35	-1.47	-1.54	-1.60	-1.61
	$(\tau_1 = 6/12, \tau_2 = 1)$	-1.50	-1.53	-1.69	-1.79	-1.86	-1.88
$\gamma = 40$	$(\tau_1 = 2/12, \tau_2 = 2)$	-0.17	-0.17	-0.19	-0.21	-0.22	-0.22
	$(\tau_1 = 6/12, \tau_2 = 1)$	-0.19	-0.19	-0.22	-0.24	-0.26	-0.26

Table 4: **Jump Exposure $\widetilde{\pi}_{q1}$ in ELW Model with Different Risk Aversions (γs).** The jump exposures $\widetilde{\pi}_{q1}$ caused by the ELW model due to model mis-specification are presented with the two risk aversions: the moderate one ($\gamma = 5$) and the extreme one ($\gamma = 40$), and the HJ model is assumed to be the true model. The typical maturity pairs of the variance swap contracts are used, as in Figure 4, including the pair with a long maturity gap (i.e., $\tau_1 = 2/12, \tau_2 = 2$) and the one with a moderate gap (i.e., $\tau_1 = 6/12, \tau_2 = 1$).

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Panel I: Economic Costs by ELW Model and JP Model for $\gamma = 5$							
Model	Maturity Pair (Year)	Investment Horizon (T) (Year)					
		0.5	1	5	10	20	30
ELW	($\tau_1 = 2/12, \tau_2 = 2$)	0.65%	1.29%	5.93%	11.23%	21.01%	29.67%
	($\tau_1 = 6/12, \tau_2 = 1$)	0.56%	1.11%	5.10%	9.73%	18.52%	26.39%
JP	($\tau_1 = 2/12, \tau_2 = 2$)	0.43%	0.86%	4.23%	8.62%	16.65%	23.55%
	($\tau_1 = 6/12, \tau_2 = 1$)	0.36%	0.72%	3.70%	8.16%	16.47%	23.33%
Panel II: Economic Costs by ELW Model and JP Model for $\gamma = 40$							
ELW	($\tau_1 = 2/12, \tau_2 = 2$)	0.54%	1.04%	4.76%	9.22%	17.64%	25.32%
	($\tau_1 = 6/12, \tau_2 = 1$)	0.53%	1.01%	4.66%	9.06%	17.53%	25.31%
JP	($\tau_1 = 2/12, \tau_2 = 2$)	0.41%	0.78%	3.56%	7.24%	14.64%	21.24%
	($\tau_1 = 6/12, \tau_2 = 1$)	0.22%	0.42%	1.97%	4.55%	11.44%	19.29%

Table 5: **Economic Costs by Model Mis-specification**. The economic costs caused by the ELW and JP model due to model mis-specification are presented, when the HJ model is assumed to be the true model. Also, the economic costs are reported with various risk aversion of the trader: the moderate risk aversion ($\gamma = 5$) and the extreme risk aversion ($\gamma = 40$). The typical maturity pairs of the variance swap contracts are used, as in Figure 4, including the pair with a long maturity gap (i.e., $\tau_1 = 2/12, \tau_2 = 2$) and the one with a moderate gap (i.e., $\tau_1 = 6/12, \tau_2 = 1$).

T (Year)	Jump Size in Variance (μ_v^P)					
	0.001	0.015	0.055	0.095	0.135	0.175
0.5	0.65%	0.67%	0.85%	1.15%	1.64%	2.29%
1	1.29%	1.42%	1.74%	2.33%	3.31%	4.68%
5	5.93%	7.56%	8.96%	11.57%	16.35%	23.87%
10	11.23%	14.84%	17.41%	22.12%	30.80%	44.06%
20	21.01%	27.72%	32.06%	39.65%	52.80%	70.22%
30	29.67%	38.59%	44.04%	53.17%	67.71%	84.03%

Table 6: **Economic Costs with Different Jump Sizes in Variance**. The economic costs that a trader with $\gamma = 5$ may suffer in ELW model due to model mis-specification are reported with a range of jump size in variance by fixing $\mu_v^Q - \mu_v^P = 0.001$. The typical maturity pair of the variance swap contracts is used, including the 2-month and 2-year variance swap contracts (i.e., $\tau_1 = 2/12, \tau_2 = 2$). Note that for each pair of (μ_v^P, μ_v^Q), HJ model is re-calibrated to the empirical mean term structure of variance swap rates reported in Table 1, which results in the RMSEs with mean 2.31% and standard deviation 0.15%.

RRA	T	4.14	4.54	4.94	**5.34**	5.74	6.14	6.54
	0.5	-	-	-	0.0000	0.0017	0.0031	0.0041
	1.0	-	-	-	0.0001	0.0035	0.0061	0.0081
$\gamma = 2$	5.0	-	-	-	0.0013	0.0179	0.0309	0.0404
	10.0	-	-	-	0.0017	0.0341	0.0591	0.0771
	20.0	-	-	-	0.0018	0.0643	0.1113	0.1445
	0.5	-	-	-	0.0000	0.0007	0.0013	0.0017
	1.0	-	-	-	0.0001	0.0015	0.0026	0.0035
$\gamma = 5$	5.0	-	-	-	0.0017	0.0090	0.0149	0.0192
	10.0	-	-	-	0.0023	0.0167	0.0281	0.0364
	20.0	-	-	-	0.0024	0.0299	0.0516	0.0674
	0.5	-	-	-	0.0000	0.0001	0.0002	0.0002
	1.0	-	-	-	0.0000	0.0002	0.0003	0.0005
$\gamma = 40$	5.0	-	-	-	0.0004	0.0014	0.0022	0.0028
	10.0	-	-	-	0.0006	0.0025	0.0041	0.0053
	20.0	-	-	-	0.0006	0.0043	0.0073	0.0096

Table 7: **Economic Costs by Mis-specifying κ_v^P in HJ Model**. The economic costs by mis-specifying κ_v^P in HJ model are calculated using the notation in Equation (22) with various risk aversion of the trader: the less risk aversion ($\gamma = 2$), the moderate risk aversion ($\gamma = 5$) and the extreme risk aversion ($\gamma = 40$). The variance swap contracts with typical maturities are used, including a set of the 2-month (τ_1), 1-year (τ_2) and 2-year (τ_3) variance swap contracts. The bold number is the true estimator for κ_v^P reported in Table 2. The symbol of "-" denotes the bankruptcy of the trader's trading position due to the negative $\tilde{\pi}_{q1}$.

RRA	T	0.371	0.411	0.451	**0.491**	0.531	0.571	0.611
	0.5	0.0022	0.0010	0.0002	0.0000	0.0005	0.0018	0.0041
	1.0	0.0040	0.0018	0.0004	0.0001	0.0012	0.0041	0.0094
$\gamma = 2$	5.0	0.0160	0.0068	0.0014	0.0013	0.0084	0.0254	0.0555
	10.0	0.0326	0.0149	0.0035	0.0017	0.0134	0.0440	0.0995
	20.0	0.0690	0.0336	0.0089	0.0018	0.0206	0.0753	0.1751
	0.5	0.0009	0.0004	0.0001	0.0000	0.0002	0.0008	0.0019
	1.0	0.0015	0.0006	0.0001	0.0001	0.0007	0.0021	0.0046
$\gamma = 5$	5.0	0.0055	0.0022	0.0007	0.0017	0.0062	0.0161	0.0339
	10.0	0.0114	0.0052	0.0017	0.0023	0.0095	0.0269	0.0604
	20.0	0.0256	0.0127	0.0039	0.0024	0.0129	0.0425	0.1025
	0.5	0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003
	1.0	0.0002	0.0001	0.0000	0.0000	0.0001	0.0003	0.0007
$\gamma = 40$	5.0	0.0006	0.0003	0.0001	0.0004	0.0012	0.0028	0.0058
	10.0	0.0013	0.0006	0.0003	0.0006	0.0018	0.0047	0.0107
	20.0	0.0030	0.0015	0.0006	0.0006	0.0023	0.0071	0.0179

Table 8: **Economic Costs by Mis-specifying κ_m^P in HJ Model**. The economic costs by mis-specifying κ_m^P in HJ model are calculated using the notation in Equation (22) with various risk aversion of the trader: the less risk aversion ($\gamma = 2$), the moderate risk aversion ($\gamma = 5$) and the extreme risk aversion ($\gamma = 40$). The variance swap contracts with typical maturities are used, including a set of the 2-month (τ_1), 1-year (τ_2) and 2-year (τ_3) variance swap contracts. The bold number is the true estimator for κ_m^P reported in Table 2.

References

Aït-Sahalia, Y., J. Cacho-Diaz, T. Hurd, 2009. Portfolio choice with jumps: a closed-form solution. Annals of Applied Probability 19, 556-584.

Aït-Sahalia, Y., Cacho-Diaz,J., Laeven, R.J.A., 2015. Modeling financial cotagion: Using mutually exciting jump processes. Journal of Financial Economics, 117(3), 585-606.

Aït-Sahalia, Y., Karaman, M., Mancini, L., 2015. The term structure of variance swap and risk premia, working paper.

Bakshi, G., Cao, C., Chen, Z., 1997. Empirical performance of alternative option pricing models. Journal of Finance, 52, 2003-2049.

Bakshi, G., and N. Kapadia. Delta-hedged gains and the negative market volatility risk premium. Review of Financial Studies, 16, 527C566.

Bajeux-Besnainou, I., Jordan, J., Portait, R., 2001. An asset allocation puzzle: a comment. American Economic Review 91 (4), 1170-1180.

Bandi, F., Reno, R., 2015. Price and volatility co-jumps. Forthcoming in Journal of Financial Economics.

Bardhan, I., X. Chao, 1996. On martingale measures when asset returns have unpredictable jumps. Stochastic Processes and their Applications 63, 35-54.

Bates, D.S., 2000. Post-'87 crash fears in the S&P 500 futures option market. Journal of Econometrics, 94, 181-238.

Bates, D.S., 2006. Maximum likelihood estimation of latent affine process. Review of Financial Studies, 19, 909-965.

Bates, D.(2008). The market for crash risk. Journal of Economic Dynamics and Control, 32, 2291C 2321.

Bollen, N. P., and R. E. Whaley (2004). Does net buying pressure affect the shape of implied volatility functions? Journal of Finance, 59, 711C753.

Bondarenko, O.(2004). Market price of variance risk and performance of hedge funds. Working Paper, University of Illinois at Chicago.

Bremaud, P., 1981. Point Processes and Queues: Martingale Dynamics, Springer, Berlin.

Broadie, M., Chernov, M. Johannes, M., 2007. Model specification and risk premia: Evidence from futures options. Journal of Finance, 62, 1453-1490.

Buraschi, A., Porchia, P. and Trojani, F.(2010). Correlation risk and optimal portfolio choice. Journal of Finance, 65(1), 393-420.

Canner, N., Mankiw, G., Weil, D., 1997. An asset allocation puzzle, American Economic Review 87 (1), 181-191.

Carr, P., and L. Wu (2009). Variance Risk Premiums, Review of Financial Studies, 22, 1311C1341.

Chernov, M., Gallant, A. R., Ghysels, E., Tauchen, G. T., 2003. Alternative models for stock price dynamics. Journal of Econometrics 116, 225-257.

Chernov, M., Ghysels, E., 2000. A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of option valuation. Journal of Financial Economics, 57, 407-458.

Cvitanic, J., I. Karatzas, 1992. Convex duality in constrained portfolio optimization. The Annals of Applied Probability 2, 767-818.

Das, S., R. Uppal, 2004. Systemic risk and international portfolio choice. Journal of Finance 59, 2809–2834.

Detemple, J., R. Garcia and M. Rindisbacher, 2003. Monte Carlo method for optimal portfolios. Journal of Finance 58, 401-406.

Duffie, D., J. Pan, K. Singleton, 2000. Transform analysis and asset pricing for affine jump-diffusions. Econometrica 68, 1343-1376.

Egloff, D., Leippold, M., Wu, L., 2010. The term structure of variance swap rates and optimal variance swap investments. Journal of Financial and Quantitative Analysis, 45, 1279-1310.

Eraker, B., Johannes, M.S., Polson, N., 2003. The impact of jumps in equity index volatility volatility and returns. Journal of Finance, 58, 1269-1300.

Eraker, B., 2004. Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices. Journal of Finance (59), 1367-1403.

Filipovié, D., Gourier, E., Mancini, L., 2015. Quadratic variance swap models. Journal of Financial Economics, forthcoming.

Fulop, A., Li, J., Yu, J., 2015. Self-Exciting Jumps, Learning, and Asset Pricing Implications. Review of Financial Studies, 28(3), 876-912.

Gârleanu, N., L. H. Pedersen and A. M. Poteshman(2009). Demand-based option pricing. Review of Financial Studies, 22, 4259-4299.

Jin, X., A. Zhang, 2012. Decomposition of optimal portfolio weights in a jump-diffusion model and its applications. Review of Financial Studies 25: 2877-2919.

Lioui, A., 2007. The asset allocation puzzle is still a puzzle, Journal of Economic Dynamic & Control 31, 1185-1216.

Liu, J., 2007. Portfolio selection in stochastic environments. Review of Financial Studies, 20, 1-39.

Liu, J., F. Longstaff, J. Pan, 2003. Dynamic asset allocation with event risk. Journal of Finance 58, 231-259.

Liu, J.; J. Pan; and T. Wang(2005). An equilibrium model of rare-event premia and its implication for option smirks. Review of Financial Studies, 18, 131C164.

Isaenko, S.(2007). Dynamic equilibrium with overpriced put options. Economic Notes, 36, 1C26.

Merton, R., 1969. Lifetime portfolio choice: the continuous-time case. Review of Economics and Statistics 51, 247-257.

Merton, R., 1971. Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory 3, 373-413.

Pan, J., 2002. The jump-risk premia implicit in options: Evidence from an integrated time-series study. Journal of Financial Economics, 63, 3-50.

Samuelson, P., 1969. Lifetime portfolio selection by dynamic stochastic programming. Review of Economics and Statistics 51, 239-246.

Sangvinatsos, A., Wachter, J.A., Does the failure of the expectations hypothesis matter for long-term investors? Journal of Finance, Vol. LX, No. 1, 179-230.

Santa-Clara, P., and A. Saretto(2009). Option Strategies: Good Deals and Margin Calls. Journal of Financial Markets, 12, 391C417.

Todorov, V., 2009. Variance risk premium dynamics: The role of jumps. Review of Financial Studies, 23, 345-383.

Zhou, G., Zhu, Y., 2012. Volatility Trading: What is the Role of the Long-Run Volatility Component? Journal of Financial and Quantitative Analysis, 47, 2012, 273–307.